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# 接触幾何学と関連分野
## 研究集会報告集

１９９５年1月１７日～２０日
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# 序

１９９４年度科学研究費補助金　総合研究(A)（代表者：西田吾郎）及び総合研究(A)（代表者：佐々木武）による研究集会「接触幾何学と関連分野」が１９９５年1月１７日から２０日まで秋田大学教育学部数学教室で開催されました。本報告はその内容を記録するものです。

研究集会はここ数年、同じテーマで、２～３時間からなるいくつかの主講演とその他１時間の通常講演から構成されてきました。今年度は、Monge-Ampere方程式とsymplectic capacityを中心的話題として、森本徹氏及び小澤哲也氏にその解説をお願いしました。さらに、佐藤肇氏には氏の最近の研究対象であるTwistor 理論について解説をしていただきました。これら三つの講演が２時間講演として実施されましたが、２時間では到底かたり尽くせる内容ではなく、参加者からぜひその解説文を書いて欲しいと言う意見が出されここに報告集を編集する事となった次第です。

研究集会初日はあの阪神大震災と重なりかなり不安を抱いた参加者（特に、関西方面の方々）も居られたようですが、成功裏に研究集会を終えることが出来たのは参加者の皆様のご協力の結果で有ります。この場を借りて皆様のご協力とさらに会場のお世話を快くお引き受け下さった、秋田大学教育学部数学教室の三上健太郎氏、川上肇氏及び小林真人氏に感謝いたします。

１９９５年　8月　7日

世話人代表：泉屋周一

# 接触幾何学と関連分野

科学研究費（総合(A) 研究代表者：西田吾郎、および総合(A) 研究代表者：佐々木武）による標記の研究集会を下記の日程で開催します.

世話人：泉屋周一（北大），伊藤敏和（龍谷大経済），佐藤肇（名古屋大），
水谷忠良（埼玉大）、小林真人（秋田大）

日時：　１９９５年1月１７日（火）～２０日（金）
場所：　秋田大学教育学部数学教室

１月１７日（火）

9:30-10:30　小澤哲也（名古屋大・理）　Symplectic capacity について I

10:45-11:45　森本徹（京都教育大）　Monge-Ampère equations の幾何 I

13:30-14:30　辻幹雄（京都産業大・理）　Monge-Ampère equations and surfaces with nonnegative Gaussian curvature

14:45-15:45　宮崎直哉（東京理科大・理工）　可逆非斉次な振動積分から構成される無限次元 Lie 群と $S^{2n-1}$上の接触変換

16:15-17:15　瀧山晃弘（北大・理）　3次元ミンコフスキー空間の中の擬球面の特徴付けについて

１月１８日（水）

9:30-10:30　森本徹（京都教育大）　Monge-Ampère equations の幾何 II

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１月２０日（金）

9:30-10:30　佐藤　肇（名古屋大・理）　接触多様体、Twistor 理論と微分方程式の幾何 II

10:45-11:45　三上健太郎（秋田大・教育）Poisson structures and foliations on 3-manifolds

13:30-14:30　松尾幸二（一関高専）　Hermitian Geometry on Hermitian manifolds

14:45-15:45　向山一男 (都立航空高専）On smooth Sp(2,R)-actions on $S^4$

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# MONGE-AMPÈRE EQUATIONS VIEWED FROM CONTACT GEOMETRY

TOHRU MORIMOTO

## Introduction

In this note I wish to give an elementary survey on Monge-Ampère equations from the view point of contact geometry. The main sources are Goursat [5], Matsuda [10], Morimoto [11], and some recent topics that I have talked on several occasions ([13] etc).

The Monge-Ampère equations, even if limited to the equations in two independent variables, are very rich in concrete examples arising from Analysis, Geometry, and Physics. On the other hand the Monge-Ampère equations are stable under contact transformation and can be well described in contact geometry.

One of the main purposes of this survey is to bring into relief various geometric problems through geometrization of the Monge-Ampère equations.

## Contents



§1. Monge-Ampère exterior differential systems.

Let us first recall the notion of exterior differential system. Let $M$ be a differential manifold and let $\mathcal{A}$ denote the sheaf of germs of differential forms on $M$. An exterior differential system on $M$ is a subsheaf $\Sigma$ of $\mathcal{A}$ such that

(1) Each stalk $\Sigma_x, x \in M$, is an ideal of $\mathcal{A}_x$, (2) $\Sigma$ is closed under exterior differentiation, i.e., $d\Sigma \subset \Sigma$, (3) $\Sigma$ is locally finitely generated.

An integral manifold of the exterior differential system $\Sigma$ is an immersed submanifold $\iota : N \to M$ such that $\iota^* \phi = 0$ for any section $\phi$ of $\Sigma$.

For detailed treatises on exterior differential systems refer to Cartan [3], Kuranishi [8], Bryant et.al [1] etc.

Now consider a contact manifold $(M, D)$ of dimension $2n + 1$. By definition, a contact structure $D$ is a subbundle of the tangent bundle $TM$ of $M$ of codimension 1 defined locally by a 1-form $\omega$ satisfying $\omega \wedge (d\omega)^n \neq 0$ (everywhere). Such a 1-form $\omega$ is called a contact form of the contact structure $D$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

**Definition.** *An exterior differential system $\Sigma$ on a contact manifold $(M, D)$ is called a Monge-Ampère exterior differential system (or simply M-A system) if $\Sigma$ is locally generated by a contact form $\omega$ of $D$ and an $n$-form $\theta$.*

By a solution of a M-A system $\Sigma$ we mean an integral manifold of $\Sigma$ of dimension $n$. Note that an integral manifold of $\Sigma$ is a fortiori an integral manifold of $D$, namely an isotropic submanifold, and a Legendre submanifold if the dimension takes the maximum value $n$. Hence a solution of a M-A system is, in particular, a Legendre submanifold.

To justify our terminology, let us see a solution of a M-A system turns out to be a solution of a so-called Monge-Ampère equation when expressed in terms of a suitable canonical coordinate system.

Let $\Sigma$ be a M-A system on a contact manifold $M$ of dimension $2n+1$ and let $\iota : S \to M$ be a Legendre submanifold. Take a point $a \in S$. By Darboux's theorem there is a local coordinate system (called a canonical coordinate system) $x^1, x^2, \cdots, x^n, z, p_1, \cdots, p_n$ of $M$ around $\iota(a)$ such that the contact structure is locally defined by the 1-form $\omega = dz - \sum_{i=1}^{n} p_i dx^i$. Moreover we can choose a canonical coordinate system so that $\iota^* dx^1, \cdots,$ $\iota^* dx^n$ are linearly independent at $a$. Then the image $\iota(V)$ may be expressed in a neighbourhood $V$ of $a$ as a graph:

$$\begin{cases} z = \phi(x^1, \cdots, x^n) \\ p_j = \psi_j(x^1, \cdots, x^n) \end{cases}$$

Since $S$ is a Legendre submanifold we have

$$\psi_j = \frac{\partial \phi}{\partial x^j}, \qquad j = 1, \cdots, n.$$

Let $\theta$ be an $n$-form which, together with $\omega$, generates the M-A system $\Sigma$ and write it down by the canonical coordinates as:

$$\theta \equiv \sum_{i_1 < \cdots < i_l, j_1 < \cdots < j_{n-l}} F_{i_1 \cdots i_l}^{j_1 \cdots j_{n-l}} dx^{i_1} \wedge \cdots \wedge dx^{i_l} \wedge dp_{j_1} \wedge \cdots \wedge dp_{j_{n-l}} \pmod{\omega.} \tag{1.1}$$

Then $\iota_{|V} : V \to M$ is a solution of $\Sigma$ if and only if

$$\sum F_{i_1 \cdots i_l}^{j_1 \cdots j_{n-l}}(x^1, \cdots, x^n, \phi, \frac{\partial \phi}{\partial x^1}, \cdots, \frac{\partial \phi}{\partial x^n}) \Delta_{j_1 \cdots j_{n-l}}^{i_1 \cdots i_l}(\phi) = 0, \tag{1.2}$$

where $\Delta_{j_1 \cdots j_{n-l}}^{i_1 \cdots i_l}(\phi)$ denotes the minor of the Hessian matrix of $\phi$ given by:

$$\begin{aligned} &\Delta_{j_1 \cdots j_{n-l}}^{i_1 \cdots i_l}(\phi) \\ &= sgn \begin{pmatrix} 1, 2, \cdots, l, l+1, \cdots, n \\ i_1, \cdots, i_l, k_1, \cdots, k_{n-l} \end{pmatrix} det \begin{pmatrix} \frac{\partial^2 \phi}{\partial x^{j_1} \partial x^{k_1}} & \cdots & \frac{\partial^2 \phi}{\partial x^{j_1} \partial x^{k_{n-l}}} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 \phi}{\partial x^{j_{n-l}} \partial x^{k_1}} & \cdots & \frac{\partial^2 \phi}{\partial x^{j_{n-l}} \partial x^{k_{n-l}}} \end{pmatrix} \end{aligned}$$

with $\{1, 2, \cdots, n\} = \{i_1, \cdots, i_l, k_1, \cdots, k_{n-l}\}$ and $k_1 < \cdots < k_{n-l}$.

A second order nonlinear partial differential equation for one unknown function $\phi$ with $n$ independent variables of the above form (1.2) is known as Monge-Ampère equation. In particular, when $n = 2$, it has the following form familiar in the classical literature (see e.g., [5]):

$$Hr + 2Ks + Lt + M + N(rt - s^2) = 0,$$

where $p = \frac{\partial\phi}{\partial x}$, $q = \frac{\partial\phi}{\partial y}$, $r = \frac{\partial^2\phi}{\partial x^2}$, $s = \frac{\partial^2\phi}{\partial x\partial y}$, $t = \frac{\partial^2\phi}{\partial y^2}$ and $H, K, \cdots, N$ are functions of $x, y, z, p, q$.

Thus a Monge-Ampère equation may be considered as a coordinate representation of a more intrinsic object of a Monge-Ampère exterior differential system.

Example 1: Consider $\mathbb{R}^5(x, y, z, p, q)$ as a contact manifold equipped with a contact form

$$\omega = dz - pdx - qdy.$$

Let $\Sigma$ be a M-A system generated by the following 2-form (and $\omega$):

$$\theta = dp \wedge dq.$$

If a solution of $\Sigma$ is represented in the form:

$$z = \phi(x, y), \quad p = \psi_1(x, y), \quad q = \psi_2(x, y),$$

then the function $\phi$ is a solution of the Monge-Ampère equation:

$$rt - s^2 = 0.$$

If we introduce another coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q})$ defined by

$$\begin{cases} x = \bar{p}, & p = -\bar{x} \\ y = \bar{y}, & q = \bar{q} \\ z = \bar{z} - \bar{p}\bar{x}, \end{cases}$$

then we have

$$\omega = d\bar{z} - \bar{p}d\bar{x} - \bar{q}d\bar{y}$$
$$\theta = -d\bar{x} \wedge d\bar{q}.$$

Hence if a solution of $\Sigma$ is represented in the form

$$\bar{z} = \bar{\phi}(\bar{x}, \bar{y}), \quad \bar{p} = \bar{\psi}_1(\bar{x}, \bar{y}), \quad \bar{q} = \bar{\psi}_2(\bar{x}, \bar{y}),$$

the function $\bar{\phi}$ satisfies the Monge-Ampr̀e equation

$$\bar{t} = 0.$$

Example 2. With the same notation as above, consider a M-A system generated by

$$\theta = dp \wedge dx.$$

Then the solutions with independent variables $x, y$ satisfy the Monge-Ampère equation $s = 0$.

If we introduce another local coordinate system defined by

$$x = \bar{z}, \quad y = \bar{x}, \quad z = \bar{y}, \quad p = \frac{1}{\bar{q}}, \quad q = \frac{\bar{p}}{\bar{q}}$$

then we have

$$\omega = -p(d\bar{z} - \bar{p}d\bar{x} - \bar{q}d\bar{y}), \quad \theta = -\frac{1}{\bar{q}^2}(\bar{p}d\bar{q} \wedge d\bar{x} + \bar{q}d\bar{q} \wedge d\bar{y}).$$

Thus the solutions with independent variables $\bar{x}, \bar{y}$ satisfy:

$$\bar{q}\bar{s} - \bar{p}\bar{t} = 0.$$

Let $(M, D)$ and $(M', D')$ be contact manifolds. A diffeomorphism $f : M \to M'$ is called a contact transformation if $f_* D = D'$. Let $\Sigma, \Sigma'$ be M-A systems on $M, M'$ respectively, we say that $\Sigma$ and $\Sigma'$ are contact equivalent (or equivalent, or isomorphic) if there exists a contact transformation $f$ such that $f^*\Sigma' = \Sigma$. The notion of "locally contact equivalent" is defined in the obvious manner.

Given a Monge-Ampère equation of the form (1.2). Associating to it a M-A system on the standard contact manifold $\mathbb{R}^{2n+1}$ generated by $n$-form $\theta$ given by (1.1), we also say that two Monge-Ampère equations are contact equivalent if so are the associated M-A systems.

The above examples show that the M-A equations $rt - s^2 = 0$ and $t = 0$ are locally contact equivalent, and so are $s = 0$ and $qs - pt = 0$.

§2. Characteristic systems.

For further geometrization of the Monge-Ampère equations, we will introduce the characteristic systems of a M-A system. It is for the M-A systems on 5-dimensional complex contact manifolds that the characteristic systems are well defined and have nice geometric properties. For this reason from now on we will work on 5-dimensional complex contact manifolds unless otherwise stated. However, most of the following discussion will remain valid in the real category under some additional assumptions.

In general, given an exterior differential system $\Sigma$ on a manifold $M$, a subspace $L$ of the tangent space $T_x M$ at $x \in M$ is called an integral element of $\Sigma$ at $x$ if for any germ of differential form $\alpha \in \Sigma_x$, the restriction of $\alpha$ to $L$ vanishes.

Now let $\Sigma$ be a M-A system on a contact manifold $M$ of dimension 5 generated by $\omega$ and $\theta$, where $\omega$ is a contact form of the contact manifold and $\theta$ is a 2-form.

For a non-zero vector $v \in T_x M$, it is clear that the line $L(v)$ generated by $v$ is an integral element of $\Sigma$ if and only if $< v, \omega >= 0$, that is, $v \in D_x$.

Now supposing that we have chosen a 1-dimensional integral element $L(v)$ with $v \in D_x$, we are looking for a 2-dimensional integral element containing it.

For $v' \in T_xM$ the plane $L(v,v')$ is an integral element of $\Sigma$ if and only if

$$\begin{cases} <v',\omega>=0 \\ <v\wedge v',\theta>=0 \\ <v\wedge v',d\omega>=0, \end{cases}$$

in other words, $v'$ is a solution of the linear equation (polar equation):

$$\begin{cases} \omega=0 \\ v\rfloor d\omega=0 \\ v\rfloor\theta=0. \end{cases}$$

The rank of this equation is 3 or 2, according to which we call $v$ regular or singular respectively. If $v$ is regular there exists a unique 2-dimensional integral element containing $v$. If $v$ is singular the 2-dimensional integral elements containing $v$ form a 1-dimensional manifold.

This being remarked, now we define the characteristic variety $\mathcal{V}(\Sigma)$ of $\Sigma$ as the union of the 1-dimensional singular integral elements:

$$\mathcal{V}(\Sigma)_x=\{v\in D_x\ ;\ v\rfloor\theta\equiv 0\pmod{\omega, v\rfloor d\omega}\}$$
$$\mathcal{V}(\Sigma)=\cup_{x\in M}\mathcal{V}(\Sigma)_x\ .$$

Then we have:

**Proposition 2.1.** *For each $x\in M$, there are following three cases to distinguish:*

*i) There exist 2-dimensional subspaces $E_x, F_x$ of $D_x$ such that*

$$\mathcal{V}(\Sigma)_x=E_x\cup F_x,\quad D_x=E_x\oplus F_x.$$

*Moreover $E_x$ and $F_x$ are perpendicular with respect to $d\omega$, i.e., $d\omega(v,v')=0$ for $v\in E_x, v'\in F_x$.*

*ii) There exists a 2-dimensional subspace $E_x$ of $D_x$ such that $\mathcal{V}(\Sigma)=E_x$ and $E_x$ is isotropic, i.e., $d\omega(v,v')=0$ for $v,v'\in E_x$.*

*iii) $\mathcal{V}(\Sigma)_x=D_x$.*

Proof. We denote by $\Omega$ and $\Theta$ respectively the restrictions of $d\omega$ and $\theta$ to $D_x$. If $\Theta+\lambda\Omega=0$ for some $\lambda\in\mathbb{C}$, then occurs the case iii), the case where the Monge-Ampère equation degenerates to be trivial at $x$.

If $\Theta\not\equiv 0$ (mod $\Omega$ ), let $\lambda_1,\lambda_2$ be the roots of the quadratic equation for $\lambda$: $(\Theta+\lambda\Omega)^2=0$. Then $\Theta+\lambda_i\Omega$ is decomposable and we can write:

$$\Theta+\lambda_1\Omega=\alpha_1\wedge\alpha_2$$
$$\Theta+\lambda_2\Omega=\beta_1\wedge\beta_2$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in D_x^*$. Let $E_x$ and $F_x$ be the null space of $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ respectively. Then we see immediately that $\mathcal{V}(\Sigma)_x = E_x \cup F_x$.

If $\lambda_1 \neq \lambda_2$, the formula

$$(\lambda_1 - \lambda_2)\Omega = \alpha_1 \wedge \alpha_2 - \beta_1 \wedge \beta_2$$

shows that $E_x$ and $F_x$ are perpendicular. Moreover since $\Omega \wedge \Omega \neq 0$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are linearly independent and hence $D_x = E_x \oplus F_x$.

If $\lambda_1 = \lambda_2$ then $E_x = F_x$ . Moreover since the equation $(\alpha_1 \wedge \alpha_2 + \mu\Omega)^2 = 0$ has only one solution $\mu = 0$, we have $\Omega \wedge \alpha_1 \wedge \alpha_2 = 0$. Then by Cartan's lemma we have

$$\Omega = \alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$$

for some $\gamma_1, \gamma_2 \in D_x^*$, which shows that $E_x$ is isotropic. q.e.d.

**Remark.** *In the real category $\mathcal{V}(\Sigma)$ does not necessarily decompose into two subspaces. If $n$ is greater than 2 the characteristic variety is in general much more complicated.*

We have thus associated to a M-A system $\Sigma$ the characteristic variety $\mathcal{V}(\Sigma)$ which, under some generic condition, decomposes into vector bundles;

$$\mathcal{V}(\Sigma) = E \cup F$$

with $E, F$ subbundles of $D$ of rank 2 satisfying $E^\perp = F$.

Conversely, given a subbundle $E$ of $D$ of rank 2, it is immediate to see that there exists a unique M-A system $\Sigma$ such that $\mathcal{V}(\Sigma) = E \cup E^\perp$.

Each bundle $E$ of $F$ is called the characteristic system associated with $\Sigma$.

Example 1. Let $\Sigma$ be the M-A system on $\mathbb{R}^5$ defined by $\theta = dp \wedge dq$. Clearly one of the characteristic systems, say $E$, is given by

$$\omega = dp = dq = 0.$$

Since $dp \wedge dq \wedge d\omega = 0$, we see that $E^\perp = E$. Therefore the other characteristic system coincides with $E$.

Example 2. Let $\Sigma$ be the M-A system on $\mathbb{R}^5$ defined by

$$\theta = dq \wedge dy + f dx \wedge dy,$$

where $f$ is a function on $R^5$. Since

$$\begin{aligned} \theta &= (dq + f dx) \wedge dy \\ &\equiv -dp \wedge dx + f dx \wedge dy \pmod{d\omega} \\ &= dx \wedge (dp + f dy), \end{aligned}$$

the characteristic systems $E, F$ are given by:

$$\begin{aligned} E &: \omega = dq + f dx = dy = 0 \\ F &: \omega = dx = dp + f dy = 0 \end{aligned}$$

§3. Monge's method of integration.

By the method of Hamilton-Jacobi one can solve any first order partial differential equation for one unknown function by integrating an ordinary differential equation, a Hamiltonian vector field. From around the turn of the 18th century much efforts have been paid to extend this method and solve the higher order partial differential equations only by integrating ordinary differential equations, where, in particular, Monge and Darboux left interesting methods. The method of Monge applies to a certain class of Monge-Ampère equations and can be found in many classical literatures in particular [5]. We notice also that M. Matsuda [9] gave a some improvement of this method. Status: RO

In this section we will give an exposition of Monge's method by making clear its geometric essence in contact geometry.

Let $\Sigma$ be a M-A system on a 5-dimensional contact manifold $M$. Let $\mathcal{V}(\Sigma)$ be the characteristic variety. According to Proposition 2.1, we write; $\mathcal{V}(\Sigma)_x = E_x \cup F_x$. To study the solutions of $\Sigma$ the following observation is fundamental:

**Proposition 3.1.** *A 2-dimensional submanifold $\iota : S \to M$ is a solution of $\Sigma$ if and only if the following conditions are satisfied:*

(1) $\iota_* T_x S \subset D_{\iota(x)}$ *for all* $x \in S$.
(2) $\iota_* T_x S \cap E_{\iota(x)} \neq 0$ *for all* $x \in S$.
(3) $\iota_* T_x S \cap E_{\iota(x)} \neq 0$ *for all* $x \in S$.

Remark: The conditions (2) and (3) above are equivalent under the condition (1).

*Proof.* As the same notation in the preceding section, let $\theta, \omega$ be local generators of $\Sigma$. If $E_{\iota(x)}$ is defined by

$$\omega_{\iota(x)} = \alpha_1 = \alpha_2 = 0$$

with $\alpha_1, \alpha_2 \in T_x^* M$, then

$$\theta_{\iota(x)} \equiv \alpha_1 \wedge \alpha_2 \pmod{\omega_{\iota(x)}, d\omega_{\iota(x)}}.$$

It follows from this that a Legendre subspace $L \subset D_{\iota(x)}$ is an integral element of $\Sigma$ if and only if $L \cap E_{\iota(x)} \neq 0$.

Now we are going to consider a Cauchy problem for $\Sigma$. Let $c : I \to M$ be a one-dimensional integral curve, that is, $c^*\omega = 0$ or $c_* T_t I \subset D_{c(t)}$ for all $t \in I$. We say the curve $c$ is non-characteristic if

$$c_* T_t I \cap \mathcal{V}(\Sigma)_{c(t)} = 0 \quad \text{for all } t \in I.$$

Given a non-characteristic curve $c$, we want to find a solution of $\Sigma$ by extending the initial curve $c$.

In view of Proposition 3.1, every solution is generated by two families of characteristic curves (integral curves of $E$ or $F$). It is, therefore, natural to expect that the surface generated by the integral curves starting from the initial curve $c$ of characteristic vector field (i.e., a section of $\mathcal{V}(\Sigma)$) would be a solution.

More precisely, let $X$ be a everywhere non-zero characteristic vector field, say, $X \in \Gamma(E)$. Let $u(t,s)$ be the integral curve of $X$ defined by

$$\begin{cases} u(t,0) = c(t), \quad t \in I \\ \dfrac{\partial u}{\partial s}(t,s) = X_{u(t,s)} \end{cases}$$

Then the map $u(t,s)$ gives an immersion $u : U \to M$, where $U$ is some neighbourhood $U$ of $I \times \{0\}$ in $I \times \mathbb{R}$.

It is clear that $u$ satisfies the condition (2) of Proposition 3.1. However it does not necessarily satisfy the condition (1) and is not a Legendre submanifold. In order that $u$ becomes a solution of $\Sigma$, we have the following criterion:

**Proposition 3.2.** *The notation being as above, if $X$ satisfies the following two conditions then $u : U \to M$ is a solution of $\Sigma$:*

(1) $< d\omega, \frac{dc}{dt} \wedge X_{c(t)} >= 0, \quad t \in I,$
(2) $L_X^2 \omega \equiv 0 \quad (\mathrm{mod}\ \omega, L_X\omega).$

*Proof.* Denote by $u_s$ the local 1-parameter transformation group generated by $X$. Since the tangent space $u_* T_{(s,t)} U$ is spanned by $\{X_{(u(t,s)}, (u_s)_* \frac{dc}{dt}(t)\}$, it suffices to prove

$$(*) \qquad \begin{cases} < (u_s)_* c'(t), \omega >= 0 \\ < (u_s)_* c'(t) \wedge X_{u(t,s)}, d\omega >= 0. \end{cases}$$

By assumption (2), we can write

$$L_X^2 \omega = a\omega + bL_X\omega$$

with some functions $a, b$ on $M$. For a fixed $t \in I$, we define functions $y_1(s), y_2(s)$ by

$$\begin{cases} y_1(s) =< (u_s)^* \omega, c'(t) > \\ y_2(s) =< (u_s)^* L_X\omega, c'(t) > . \end{cases}$$

Then $y_1, y_2$ satisfy the following ordinary differential equations:

$$\begin{cases} \dfrac{dy_1}{ds} = y_2 \\ \dfrac{dy_2}{ds} = ay_1 + by_2. \end{cases}$$

But $y_1(0) = y_2(0) = 0$ by the assumption (1). It then follows from the uniqueness of solution that $y_1 = y_2 = 0$, which proves (*).

A non-zero vector field $X \in \Gamma(E)$ is called an integral characteristic vector field of $E$ if the condition (2) of Proposition 3.2 is satisfied, and further it is called adapted to the initial curve $c$ if the condition (1) is satisfied.

Now let us consider how to find an adapted integrable characteristic vector field. We assume the characteristic variety decomposes into vector bundles: $\mathcal{V}(\Sigma) = E \cup F$, so that $E^{\perp} = F$.

Now suppose that there exists a first integral $h$ of $F$, that is, $Yh = 0$ for any $Y \in \Gamma(F)$. Let $X_h$ be the vector field defined by

$$\text{(3-1)} \qquad \begin{cases} < X_h, \omega >= 0 \\ L_{X_h}\omega \equiv dh \pmod{\omega}. \end{cases}$$

Then we see firstly that $X_h \in \Gamma(E)$, because $d\omega(X_h, Y) =< dh, Y >= 0$ for all $Y \in \Gamma(F)$. Secondly we see that $X_h$ is integrable, because

$$\begin{aligned} L^2_{X_h}\omega &= L_{X_h}(dh + \lambda\omega) \\ &\equiv L_{X_h}(dh) \pmod{\omega, L_X\omega} \\ &= d(X_h \cdot h) \\ &= d(X_h \wedge X_h, d\omega) \\ &= 0. \end{aligned}$$

Thus a first integral $h$ of $F$ gives rise to an integral characteristic vector field $X_h$ of $E$.

Next suppose that there exist two independent first integrals $h_1, h_2$ of $F$. We assume moreover $(dh_1)_x, (dh_2)_x, \omega_x$ are independent everywhere. (Since any contact form cannot be written as $\omega = \lambda_1 h_1 + \lambda_2 h_2$, this condition is satisfied for generic points.)

Then for any non-characteristic integral curve $c$ of $\Sigma$, we can locally find an integral characteristic vector field $X_h$ adapted to $c$ as follows: For $t_0 \in I$, one of $< dh_i, \frac{dc}{dt}(t_0) >$ $(i = 1, 2)$ is not zero (otherwise $c_* T_{t_0} I \subset F_{c(t_0)}$). Hence there exists a local function $h(h_1, h_2)$ such that $h(c(t)) = 0$ and $\{dh, \omega\}$ are independent. Then, since $h$ is constant on the curve $c$ we have

$$< c'(t) \wedge (X_h)_{c(t)}, d\omega >=< c'(t), dh >= 0,$$

which shows that $X_h$ is adapted to $c$. By integrating $X_h$, we obtain a solution $u : U \to M$ of $\Sigma$ with initial curve $c$.

It should be noted that $h \equiv 0$ on any solution $S$ of $\Sigma$ with initial curve $c$. In fact, since $F$ defines on $S$ a one-dimensional foliation transversal to $c$ and $h = 0$ on $c$ and $h$ is constant along the foliation, $h$ must be identically zero on $S$. This implies that any solution with initial curve $c$ is also a solution of the first order partial differential equation $h = 0$ (recall that a function on a contact manifold may be regarded as a first order PDE).

It also immediately follows that a solution of $\Sigma$ with initial curve $c$ is unique as a germ of submanifold.

Summarizing the above discussion, we have:

**Theorem 3.3.** *If one of the characteristic systems of a M-A system admits independent two first integrals, then any non-characteristic integral curve can be extended to a solution* *of the M-A system. This solution is locally unique and can be obtained by integrating an adapted integrable characteristic vector field.*

This is the integrating method of Monge. The function (or the first order PDE) $h$ is called *integrale intermediate.*

§4. Classification.

In this section we treat the problem of classifying the M-A systems vis-à-vis the local contact equivalence. Since there are infinitely many different classes and the complete classification is far from expected, here we will be content to give an outline of the classification for a rough grasp of the variety of M-A systems and add some indication for more detailed classification.

First of all note that two M-A systems $\Sigma$ and $\Sigma'$ on contact manifolds $M$ and $M'$ respectively are contact equivalent by a contact transformation $\phi : M \to M'$ if and only if $\phi_* \mathcal{V}(\Sigma) = \mathcal{V}(\Sigma')$, which is, in turn, equivalent to saying that $\phi_* E = E'$ (or $\phi_* E = F'$ ) when the characteristic varieties are decomposed into vector bundles: $\mathcal{V}(\Sigma) = E \cup F$, $\mathcal{V}(\Sigma') = E' \cup F'$. Thus the classification of the M=A systems generically reduces to classifying the pairs $(D, E)$, where $D$ is a contact structure of rank 4 and $E$ is a subbundle of rank 2 of $D$.

Here we recall the work of É. Cartan [2], in which he investigated the classification of the Pfaff equations

$$\omega_1 = \omega_2 = \omega_3 = 0$$

on a five dimensional space, in other words the subbundles of rank 2 of the tangent bundle of a five-dimensional manifold.

According to N. Tanaka [16], given a subbundle $E$ of the tangent bundle $TM$ of a manifold $M$, we define the derived systems of $E$ as follows: Define inductively the subsheaves $\mathcal{E}^k (k = 1, 2, \cdots)$ of the sheaf $\underline{TM}$ of the germs of vector fields on $M$ by setting $\mathcal{E}^1 = \underline{E}$, the sheaf of the germs of sections of $E$ and

$$\mathcal{E}^{k+1} = \mathcal{E}^k + [\mathcal{E}^1, \mathcal{E}^k].$$

Then in a neighbourhood of a generic point all $\mathcal{E}^k$ are vector bundles, that is, there exist subbundles $E^k$ such that $\mathcal{E}^k = \underline{E^k}$ $(k = 1, 2, \cdots)$.

Now in the case $\dim M = 5$ and $\operatorname{rank} E = 2$, if the derived systems are all vector bundles, there are the following five cases to distinguish:

(0) $\operatorname{rank} E^2 = 2$
(1) $\operatorname{rank} E^2 = \operatorname{rank} E^3 = 3$
(2) $\operatorname{rank} E^2 = 3, \ \operatorname{rank} E^3 = \operatorname{rank} E^4 = 4$
(3) $\operatorname{rank} E^2 = 3, \ \operatorname{rank} E^3 = 4, \ \operatorname{rank} E^4 = 5$
(4) $\operatorname{rank} E^2 = 3, \ \operatorname{rank} E^3 = 5.$

For a M-A system $\Sigma$ with characteristic variety $\mathcal{V}(\Sigma) = E \cup F$, We say that $\Sigma$ is hyperbolic if $E_x \neq F_x$ for all $x$, and parabolic if $E = F$.

Furthermore we say that a hyperbolic M-A system $\Sigma$ is in the class $H_{ij}$ if $E$ is of type (i) and $F$ of type (j) in the list above. Since there is no canonical way to distinguish $E$

and $F$ we may assume $i \leq j$. We say that a parabolic M-A system $\Sigma$ is in the class $P_j$ if $E$ is of type (j).

It should be noted that the classes $H_{ij}$ and $P_j$ are invariant under contact equivalence.

**Proposition 4.1.** *If one of the characteristic systems of a M-A system is completely integrable then the two characteristic systems coincide. Moreover such M-A systems are all locally contact equivalent.*

*Proof.* Let $\Sigma$ be a M-A system and assume that one of the characteristic systems, say $E$ is completely integrable. For $u, v \in E_x$, take a local sections $X, Y$ of $E$ such that $X_x = u, Y_x = v$. Then we have

$$\begin{aligned} d\omega(u,v) &= d\omega(X,Y)_x \\ &= [X\omega(X) - Y\omega(X) - \omega([X,Y])]_x \\ &= -\omega([X,Y])_X \\ &= 0, \end{aligned}$$

which shows that $E_x$ is isotropic and therefore $E_x^{\perp} = E_x$. Hence the two characteristic systems coincide.

To prove the last half of the assertion, we first note that if $E$ is completely integrable then we have locally a fibring $\pi : M \to X$ with the fibres being leaves of $E$ and therefore Legendre submanifolds of $M$. Then our assertion follows from the fact that Legendre fibrings are all locally contact equivalent, and this fact can be shown as follows: Let $\pi : M \to X$ be a Legendre fibring with $M$ $(2n+1)$-dimensional manifold equipped with a contact structure $D$, so that $X$ is $n+1$-dimensional and each fibre is Legendre submanifold. Let $Gr(X, n) \to X$ be the Grassmann bundle whose fibre at $x \in X$ consists of all $n$-dimensional subspaces of $T_xX$. Then there is a canonical map $f : M \to Gr(X, n)$ defined by $f(p) = (\pi(p), \pi_* D_p)$ for $p \in M$. It is easy to see that $Gr(X, n)$ has a canonical contact structure and that $f$ is a fibre preserving contact immersion. It then follows that two Legendre fibrings of same dimension are locally contact equivalent. q.e.d.

An example is fulfilled by a Monge-Ampère equation $rt - s^2 = 0$. The corresponding M-A system is generated by a 2-form $dp \wedge dq$. One of its characteristic system is given by:

$$\omega = dp = dq = 0,$$

which is clearly completely integrable. The above proposition shows that there is no M-A system in the class $H_{0j}$ and that there is only one (up to local contact equivalence) in the class $P_0$.

**Theorem 4.2.** *There is only one M-A system up to local contact equivalence in the class $H_{11}$. In other words, let $\Sigma$ be a hyperbolic M-A system with characteristic systems $E, F$. If the derived systems $E^2, F^2$ respectively of $E, F$ are both completely integrable, then $\Sigma$ is locally contact equivalent to the M-A system corresponding to the equation $s = 0$.*

This theorem goes back to S. Lie. For a classical proof see Goursat [5], or Matsuda [9]. Another proof based on the general method for the equivalence problems of geometric structures can be found in Morimoto [14].

Let us just see that the equation $s=0$ is in fact in $H_{11}$. The corresponding M-A system is defined by the 2-form

$$dq \wedge dy \equiv -dp \wedge dx \quad (\bmod \omega, d\omega).$$

Hence the characteristic systems are given by

$$\begin{aligned} E &: \omega = dq = dy = 0 \\ F &: \omega = dp = dx = 0 \end{aligned}$$

Their derived systems $E^2, F^2$ are given by

$$\begin{aligned} E^2 &: dq = dy = 0 \\ F^2 &: dp = dx = 0, \end{aligned}$$

which are clearly completely integrable.

We can also find in the book of Goursat [5] the following propositions:

**Proposition 4.3.** *If a M-A system is hyperbolic and each of its characteristic system admits a first integral, then the M-A system is locally equivalent to a Monge-Ampère equation of the following form:*

$$s + f(x, y, z, p, q) = 0$$

*Proof.* Let $E$, $F$ be the characteristic systems of a hyperbolic M-A system $\Sigma$ and assume that they have first integrals $x$, $y$ respectively. Since $E \cap F = 0$, $dx, dy$ are linearly independent. In general on a contact manifold with a fixed contact form $\omega$, we define the bracket $[f,\ g]$ for functions $f$, $g$ by:

$$[f,\ g] = d\omega(X_f,\ X_g),$$

where the vector field $X_f$ is given by (3.1). Then we see that $[x,\ y] = 0$, since $X_x$, $X_y$ are sections of $F$, $E$ respectively and therefore perpendicular. It then follows from a fundamental theorem of contact geometry that the functions $x$, $y$ can be extended to a normal coordinate system $x, y, z, p, q$. Let $E$ be defined by the Pfaff equation:

$$dx = \alpha = \omega = 0,$$

where we may suppose that $\alpha$ is written as:

$$\alpha = \alpha_2 dy + \alpha_3 dp + \alpha_4 dq.$$

But since $d\omega(X_\alpha,\ X_y) = 0$, we see $\alpha_4 = 0$. Moreover we can easily see that $\alpha_3 \neq 0$. Hence we may choose $\alpha$ as

$$\alpha = f dy + dp.$$

Thus our M-A system is generated by the 2-form:

$$dx \wedge (dp + f dy),$$

which is equivalent to the Monge-Ampère equation:

$$s + f = 0.$$

**Proposition 4.4.** *If a M-A system is parabolic and its characteristic system admits a first integral, then the M-A system is locally equivalent to a Monge-Ampère equation of the following form:*

$$t + f(x, y, z, p, q) = 0$$

*Proof.* Similar to the proof of Proposition 4.3.

The discussions above provide us with a rough idea of classifying the M-A systems. Now we give some remarks and indications for further detailed classification.

1) $H_{ij}$ $(1 \leq i \leq j \leq 4)$ is not empty: In fact each $H_{ij}$ contains infinite number of different equivalence classes except that $H_{11}$ does only one.

2) $P_0$ contains only one equivalence class, $P_1$ is empty, and $P_j$ $(j = 2, 3, 4)$ contains infinite number of classes.

3) The M-A systems in $H_{1j}$ are Monge integrable.

4) A M-A system may be treated as a $G$-structure on a contact manifold (or generalized $G$-structure on a filtered manifold as developed in Morimoto [14]) and we can apply the general methods of equivalence for (generalized) $G$-structures to obtain further invariants of M-A systems. However, since the classification spreads into many branches, it is difficult to carry out all the calculation. One of the goals which may be attainable is to classify the homogeneous M-A systems in each $H_{ij}$ or $P_j$. (We say a M-A system $\Sigma$ is homogeneous if the automorphism group $\mathrm{Aut}(\Sigma)$ of contact transformations is transitive.) (see [11]).

5) Most generic is a M-A system $\Sigma$ belonging to $H_{44}$, which satisfies:

$$\dim \mathrm{Aut}\ (\Sigma) \leq 8,$$

and if the equality holds then it is locally equivalent to a M-A system defined on the homogeneous space $SL(3, \mathbb{C})/SL(2, \mathbb{C})$ and invariant by the actions of $SL(3, \mathbb{C})$. In a suitable local expression it can be written in the following form:

$$rt - s^2 = (z - xp - yq)^4.$$

See [11],[12], and for particular solutions of this equation see A. Kushner and B. Doubrov [4].

§5. Global solutions, singularities

So far we have been mainly concerned with local problems. But since our formulation of Monge-Ampère equations is quite free from coordinate expressions, we are ready to consider various global problems as well as singularities of solutions. Here we will touch upon some of such problems.

5.1. A global model of the equation $rt - s^2 = 0$.

Let $V$ be a 4-dimensional vector space and $V^*$ its dual space. Denote by $P(V)$ and $P(V^*)$ be the projective spaces of the 1-dimensional subspaces of $V$ and $V^*$ respectively. Set

$$Q = \{([x], [\xi]) \in P(V) \times P(V^*) : \langle x, \xi \rangle = 0\}$$

With the canonical projections $\rho : Q \to P(V)$ and $\rho' : Q \to P(V^*)$, we can identify $Q$ with the projective cotangent bundle of $P(V)$ and that of $P(V^*)$:

$$Q \cong PT^*P(V) \cong PT^*P(V^*).$$

Moreover there is a canonical contact structure $D$ on $Q$ such that the above isomorphisms are contact isomorphisms. We then have

$$D = Ker\rho_* \oplus Ker\rho'_*.$$

The subbundle $Ker\rho'_*$ of $D$ of rank 2 then defines a M-A system that has $Ker\rho'_*$ as a characteristic system; we denote by $\Sigma_0$ the M-A system on $Q$ thus defined. Since $Ker\rho'_*$ is completely integrable, the M-A system $\Sigma_0$ is in the class $P_0$ and locally isomorphic to $rt - s^2 = 0$ by Proposition 4.1. It should be remarked that this M-A system is canonically associated with projective geometry.

By Proposition 3.1, a surface $S$ in $Q$ is a solution of $\Sigma_0$ if and only if $S$ is a Legendre submanifold and the rank of $\rho' : S \to P(V^*)$ is less or equal to 1 everywhere. We may consider $\Sigma_0$ as a Monge-Ampère equation for a surface of $P(V)$; a surface $Y \subset P(V)$ is called a solution if the Legendre lift $\tilde{Y}$ (in other words, the projective cotangent lift, or the projective conormal bundle of $Y$) is a solution of $\Sigma_0$. It then turns out that a solution $Y \subset P(V)$ is a ruled surface generated by projective lines and the tangent spaces of $Y$ are constant along each generating line. For instance the cone $C$ defined by

$$z^2 = x^2 + y^2$$

with affine coordinates $x, y, z$ is a solution of $\Sigma_0$. We note that the projective cotangent lift $\tilde{C}$ of $C$ is diffeomorphic to a torus and has no singularity, while $C$ does. As to the non-singular global solutions, we have the following:

**Theorem 5.1.** *A compact connected smooth surface of $P(V)$ is a solution of $\Sigma_0$ if and only if it is a projective plane.*

We remark an interesting contrast between this theorem and that of Bernstein which asserts that a solution defined on the whole $xy$-plane of the equation $rt - s^2 = 1$ is a polynomial of degree 2. The latter arises from the ellipticity of the equation, but the former arises from some properties of projective geometry and holds both in the real and the complex category.

If $S \subset Q$ is a compact connected smooth solution of $\Sigma_0$ and not the Legendre lift of a projective plane, then, by the above theorem, the projection $\rho : S \to P(V)$ must have always singularities. Here we ask a question: Is there any law for the number of singularities with respect to the projection $\rho : S \to P(V)$?

5.2. A global model of $s = 0$.

As we have seen in our classification, there is another local trivial Monge-Ampère equation, $s = 0$. Let us present a global model of this equation.

Let $\pi : S^3 \to S^2$ be the Hopf fibring, that is,

$$S^3 = \{(z_1, z_2) \in \mathbb{C}; |z_1|^2 + |z_2|^2 = 1\},$$

and $\pi$ is the quotient map to $S^2 = \mathbb{C}P^1$ by the $S^1$-action:

$$(z_1, z_2)e^{i\theta} = (e^{i\theta}z_1, e^{i\theta}z_2).$$

Then there is a natural contact form $\omega$ on $S^3$ given by the restriction of the form

$$\frac{1}{2}\mathcal{I}m(\bar{z}_1 dz_1 + \bar{z}_2 dz_2),$$

which is invariant by the $S^1$-action.

Now preparing two copies of the Hopf fibring, $\pi : S^3 \to S^2$ and $\bar{\pi} : \bar{S}^3 \to \bar{S}^2$, we set

$$M = S^3 \times \bar{S}^3 / \sim,$$

the quotient space by the action of the diagonal $\Delta$ of $S^1 \times \bar{S}^1$. Then we see that the 1-form $\omega - \bar{\omega}$ induces a contact structure $D$ on $M$. We have also the natural projections $\rho : M \to S^2$ and $\bar{\rho} : M \to \bar{S}^2$. If we set

$$E = D \cap Ker\rho_*, \quad \bar{E} = D \cap Ker\bar{\rho}_*,$$

then we see that $E$ and $\bar{E}$ are of rank 2 and perpendicular and we have

$$D = E \oplus \bar{E}.$$

Hence there is a unique M-A system $\Sigma_{11}$ on $M$ that has $E$ and $\bar{E}$ as characteristic systems. Since $E$ and $\bar{E}$ have respectively two independent first integrals given by $\rho$ and $\bar{\rho}$, the M-A system $\Sigma_{11}$ is locally equivalent to $s = 0$ by Theorem 4.2.

It would be interesting to study the global solutions of this M-A system.

5.3. Singularities of solutions.

When we study singularities of a solution $S$ of a M-A system $\Sigma$ on a contact manifold $M$, we should distinguish the following different sorts of singularities:

(a) Singularities of $S$ itself.

(b) Singularities with respect to a Legendre fibring: If there is given a Legendre fibring $\rho : M \to N$, in particular, if $M$ is the projective cotangent bundle $PT^*N$ of a manifold $N$, it would be interesting to study the singularities of $S$ with respect to the projection $\rho : S \to N$.

(c) Singularities arising with respect to a prescribed space of independent variables. If $M = J^1_X N$, that is $M$ is the space of the 1-jets of cross-sections of a fibred manifold $N \to X$, the singularities with respect to the projection $S \to X$ are the singularities which arise when one want to regard $S$ as an ordinary solution of the Monge-Ampère equation expressed in coordinate with independent variables in $X$.

Singularities of Legendre varieties have been studied by many peoples. Now, in relation with Monge-Ampère equations, we pose the following general questions:

(1) What happens to the singularities of a Legendre variety if imposed to be a solution of a M-A system?

(2) How does it depend on the choice of a M-A system?

Here we mention some result obtained by G. Ishikawa concerning to the questions above. Consider a map-germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^5, 0$ defined by:

$$(x, y, z; p, q) \circ f = (u, v^2, uv^3; v^3, \frac{3}{2}uv),$$

which is isotropic with respect to the contact form $\omega = dz - pdx - qdy$, that is, $f^*\omega = 0$. An isotropic map-germ $g$ from $\mathbb{R}^2, 0$ to a contact manifold $M$ is called an open umbrella if it is contact equivalent to the above map-germ $f$ up to parametrization.

**Proposition 5.2.** *An open umbrella can be a solution of a M-A system of type* $rt - s^2 = 0$ *but cannot be a solution of a M-A system of type* $s = 0$

For further information on the singularities of a solution of $\Sigma_0$, we refer to Ishikawa ([6],[7]). We cite also another approach of M. Tsuji [17] to singularities of Monge-Ampère equations.

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# Symplectic Capacity について

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はじめに。

このノートの目的は Hofer-Zehnder により定義された symplectic capacity について概説することにあります。

symplectic 幾何学は、もともと Hamilton 力学系の理論をわかりやすく記述することからはじまりました。Poisson 積、Lagrange 部分多様体、momentum 写像、Maslov 指数などこの流れの中で整備されてきた多くの概念があります。

力学系の理論の中では周期解の存在は中心的問題でしたから、当然 symplectic 幾何としてもこれに関連する問題は、多く

の人たちに興味をもたれましたし、その中から洗練された予想（Weinstein, Arnold, ---）もうまれ，美しい形で解かれたものもあります。

一方 symplectic 幾何が幾何学として1人歩きをはじめれば自然におこる問題があります，すなわち分類と不変量に関する問題です（何かが定ぎされればすぐにそれらを分類したがるのは数学者にありがちな安易な発想ですが、その試みが例えばよい不変量と結びついて本質がしだいに見えてくることもよくあることです）。その原形ともいえる問題として次が考えられます；「$\mathbb{R}^{2n}$の標準的な symplectic 構造に関し，2つの open sets はいつ symplectic 同相型か？」 symplectic 構造の定ぎを聞けばだれでも考えたくなる問題ですが（きっと Darboux も考えたんだろう），これに対しはじめて

symplectic 十力として本質的な break through をはたしたのは Gromov です（彼が開発した pseudo-holomorphic curve を用いる方法の応用として）。彼は次を示しました；

「$0 < r < R$ のとき，半径 $R$ の $2n$ 次元 ball $B^{2n}(R)$ を半径 $r$ の cylinder $B^2(r) \times \mathbb{R}^{2n-2}$ に symplectic に埋めこむことはできない」 (Squeezing thm)。

対称2次形式を歪対称におきかえ（+ "$d\omega = 0$"）れば Riemann 幾何は symplectic 幾何に変わりますが，Riemann にくらべ symplectic はかなりやわらかい幾何です。上の Squeezing thm （及びその発展型である symplectic packing に関するいろいろな結果など）からそのやわらかさの程度を少し知ることができます。

Squeezing thm にしげきを受けた人たちによって周期解存在問題の道具を剛柔問題(あるいは分類・不変量問題)に応用する試みとともに symplectic capacity という概念が生まれました。Symplectic 幾何を理解するうえでよい目安になるものだと思います。

$\mathbb{R}^{2n}$ の open set (あるいは一般の symplectic 多様体)の symplectic 不変量の中で、ある種の性質をみたすものを総じて symplectic capacity とよんでいます。具体例として、上の squeezing thm の直接の応用として定義される symplectic width があげられます。周期解のことばを使っていく種類かの symplectic capacity が構成されていますが、このノートでは Hofer と Zehnder が作った capacity (これを以下 $c_0$ と書く)を主に話をします。

$\mathbb{R}^{2n}$を，標準的symplectic form $\omega_0 = \sum dp_i \wedge dq_i$により，symplectic多様体と考えれば，$C^\infty$-関数 $f : \mathbb{R}^{2n} \to \mathbb{R}$ に対しHamilton方程式 $\dot{x} = X_f(x)$ が決まります。ここで $X_f = \sum \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right)$。この方程式がもし周期$= t_0$の周期解をもったとすると、$f$をscaler培した~~こと~~もの$af$ $(a \in \mathbb{R})$に対するHamilton方程式 $\dot{x} = X_{af} = aX_f$ もこれを周期解としてもち，その周期は $t_0/a$ となります。$\mathbb{R}^{2n}$のopen set $U$に対し，$\mathrm{supp}\, f \subset U$ なる $f \in C^\infty(\mathbb{R}^{2n})$で，対応するHamilton方程式のすべての周期解の周期が1以下にならないようにし$f$の振幅 $\max(f) - \min(f)$ をできるだけ大きくなるように$f$を選びなおしていったとき，この振幅のsupが$c_0(U)$です。

この関数 $c_0$ が capacity とよぶにふさわしいものあることを保証するのが次の3つの性質です；A1) $\exists \varphi : U \hookrightarrow V$ s.t. $\varphi^* \omega_0 = \alpha \omega_0$ with $\alpha \in \mathbb{R} - \{0\} \Rightarrow c_0(U) \leq |\alpha| c_0(V)$，A2) 半径＝1の $2n$次元 ball $B^{2n}(1)$ に対し $c_0(B^{2n}(1)) = \pi$，A3) 半径1の cylinder $B^2(1) \times \mathbb{R}^{2n-2}$ に対し $c_0(B^2(1) \times \mathbb{R}^{2n-2}) = \pi$。 A1 と不等式 $c_0(B^{2n}(1)) \geq \pi$ はほとんど明らかですが，$c_0(B^2(1) \times \mathbb{R}^{2n-2}) \leq \pi$ は non trivial です。

$\mathbb{R}^{2n}$ の，support が compact な symplectic diffeom. の全体の群上に定まるある自然な距離関数があります。この群の接 vector は，$\mathbb{R}^{2n}$ の support が cpt な vector 場 $X$ であって，$X$ による $\omega_0$ の Lie 微分 $L_X \omega_0$ が消えものと同一視でき，$\mathbb{R}^{2n}$ は topology が簡単なので このような vector 場は ある一意的に定まる support が cpt な関数 $f : \mathbb{R}^{2n} \to \mathbb{R}$ の Hamilton vector 場

$X = X_f$ になるので，$\|X\| := \max(f) - \min(f)$ として norm が決まり、従って上の群に擬距離が定まりますが（2つの symplectic diffeom が smooth path で結べないときは そのキョリは $\infty$），実はこれは距離になります。

この距離を使って open set $U \subset \mathbb{R}^{2n}$ のくみ出し energy $e(U)$ を次のように定義します；$\psi(U) \cap U = \phi$ となる symplectic diffeom $\psi$ の中で $id_{\mathbb{R}^{2n}}$ に一番近いものの距離 (inf) ~~として~~ を $e(U)$ とする。このとき不等式

$$c_0(U) \leq e(U)$$

が成り立ちます。cylinder のくみ出し energy $e(B^2(1) \times \mathbb{R}^{2n-2})$ が $\pi$ 以下であることはすぐにわかるので この不等式から 先の不等式 $c_0(B^2(1) \times \mathbb{R}^{2n-2}) \leq \pi$ が得られます。

Hofer-Zehnder の本 [1] は この方面のよい解説書で，このノートのほとんどは これを参考にしています。

## Hamilton方程式と変分原理：

$\mathbb{R}^{2n}$の座標を$(p_1 \cdots p_n q_1 \cdots q_n)$とおき複素構造$J$を$J(p_1 \cdots q_n) = (-q_1 \cdots -q_n\, p_1 \cdots p_n)$とすると標準symplectic form $\omega_0 = \sum dp_i \wedge dq_i$は$\omega(X, Y) = \langle JX, Y \rangle$，ただし$\langle\ ,\ \rangle$は$\mathbb{R}^{2n}$のEuclid内積。関数$H \in C^\infty(\mathbb{R}^{2n})$に対応するHamilton方程式は

$$\dot{x} = X_H = -J\nabla_H$$

ただし$\nabla_H$は$H$の gradient vector場。

$X_H$が生成する1-parameter群$\exp tX_H$を$\varphi_H(t)$で表わし，$H$のHamilton流という。$H$が時間に依存するとき$H \in C^\infty([0,1] \times \mathbb{R}^{2n})$も$H_t(x) = H(t, x)$とおき方程式$\dot{x} = X_{H_t}(x)$と考え対応するHamilton流を同じ記号$\varphi_H(t)$で表わす。このとき$\varphi_H(t) \in \mathrm{Diff}(\mathbb{R}^{2n})$はsymplecticです；$\varphi_H(t)^* \omega_0 = \omega_0$．

周期が$T \in \mathbb{R}_{>0}$の閉曲線$x \in C^\infty(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$

に対する汎関数 $a_H(x)$ を

$$a_H(x) := \int_0^T -\frac{1}{2}\langle J\dot{x}, x\rangle - H(t, x(t))\, dt$$

とおけば $a_H$ の Euler-Lagrange 方程式は先の Hamilton 方程式と一致します。もし $H$ が $t$-不変のとき Hamilton 方程式の解 $x$ は $H = H(x(0))$ で定まる超曲面 $S$ 上にとどまる。そこで上の汎関数 $a_H$ を $C^\infty(\mathbb{R}/T\mathbb{Z}, S)$ に制限したものを考えればよいわけだが、これは

$$a(x) = \int_0^T -\frac{1}{2}\langle J\dot{x}, x\rangle dt$$

とおいて $a : C^\infty(\mathbb{R}/T\mathbb{Z}, S) \to \mathbb{R}$ を考えることと同じだから、$a$ のそくばく条件つき Euler-Lagrange 方程式はやはりもとの Hamilton 方程式と一致する。

Hofer-Zehnder の symplectic capacity：

対応 $c : \{\mathbb{R}^{2n} \text{の open sets}\} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ が次をみたすとき これを symplectic capacity とよびます；

A1) $\exists \varphi : U \hookrightarrow V$ ; $C^\infty$-map

$\quad$ s.t. $\varphi^* \omega_0 = \alpha \omega_0$ $\quad (\alpha \in \mathbb{R} - \{0\})$

$$\Rightarrow \quad c(U) \leq |\alpha| \, c(V)$$

A2) $\quad c(B^{2n}(1)) > 0$

A3) $\quad c(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$

ただし $B^{2m}(r) \subset \mathbb{R}^{2m}$ は半径 $r$ の $2m$ 次元 ball。

(A1) より $U$ と $V$ とが symplectic 同型なら $c(U) = c(V)$ となるので symplectic capacity は symplectic invariant です。

$$c(U) := \sqrt[n]{\int_U \omega^n} \quad \text{は A1, A2 をみたし}$$

ますが A3 はみたしません。

$$c(U) := \sup\{r \mid \exists\varphi : B^{2n}(r) \hookrightarrow U : \text{symplectic embedding}\}$$

とおくと、Gromov の squeezing thm より、これは symplectic capacity の例になっています。これは symplectic width とよばれています。

与えられた $\mathbb{R}^{2n}$ の open set $U$ 上の $C^\infty$-関数 $f$ に関する次の条件を考えます；

i) $f \leq 0$

ii) $\exists K \subset U$ : cpt s.t. $f|_{U \setminus K} \equiv 0$

iii) $f$ に対する Hamilton 方程式 $\dot{x} = X_f(x)$ の定値でない周期解の周期は 1 以上。

(i), (ii) をみたす $C^\infty$-関数全体を $\mathcal{A}(U)$ 更に (iii) をみたすもの全体を $\mathcal{A}_1(U)$ とおきます。

Hofer-Zehnder の capacity $c_0$ は次のように定義されます；

$$c_0(U) := \sup_{f \in \mathcal{A}_1(U)} \{-\min(f)\}$$

この関数 $c_0$ が capacity の axiom A1 をみたすことはほとんど自明です。また $c_0(B^{2n}(1)) \geq \pi$ であることは $f(x) := \pi(1-|x|^2)$ を少し modify すればすぐに確められます。A3 をみたすことは non trivial な結果で、Hofer-Zehnder は、$f \in \mathcal{A}(B^2(1) \times \mathbb{R}^{2n-2})$ の最小値が $-\pi$ より小さいとき $f$ の Hamilton 方程式は必ず周期が 1 の周期解をもつことを証明しました。上のことと合わせて

$$\pi \leq c_0(B^{2n}(1)) \leq c_0(B^2(1) \times \mathbb{R}^{2n-2}) \leq \pi$$

ということになり、従って $c_0$ が symplectic

capacity であることがわかります。後で くみ出しenergy と この capacity $c_0$ との関係について考えるとき この不等式
$c_0(B^2(1)\times\mathbb{R}^{2n-2}) \le \pi$ について再度議論します。

symplectic capacity $c_0$ は $\mathbb{R}^{2n}$ の open set に対してだけでなく一般の symplectic 多様体 $(M,\omega)$ に対しても定義できます。ただし $\partial M \neq \phi$ のときは $\mathcal{A}(M)$ の条件 (ii) にでてくる compact set $K$ は $M$ の内部 $M\setminus\partial M$ に含まれることを要求します。

$n=1$ のとき すなわち 2次元 symplectic 多様体 $(M,\omega)$ に対しては

$$c_0(M,\omega)=\left|\int_M \omega\right|$$

であることが容易にわかります。

関数 $f$ に対応する Hamilton 方程式 $\dot{x}=X_f(x)$ の解に沿って $f$ は常に const. であることに注意します。symplectic 多様体 $(M,\omega)$ の Hamilton 方程式 $\dot{x}=X_f(x)$ の定数でない周期解で level surface $f^{-1}(l)$ 上にあるもの全体を $Per_l(f)$ とおきます。もし関数 $\varphi:\mathbb{R}\to\mathbb{R}$ : $C^\infty$ で、各 $l\in f(M)$ に対し

$$(x\text{の周期})\geq\varphi(l)>0\quad\forall x\in Per_l(f)$$

となるものが存在すれば

$$c_0(M,\omega)\geq\int_{\min f}^{0}\varphi(l)\,dl$$

となります。($f\in\mathcal{A}(M)$。) 実際

$$\rho(t):=\int_{t}^{0}-\varphi(l)\,dl$$ とおけば

$\rho\circ f\in\mathcal{A}_1(M,\omega)$ となります。

特に，もし$(M,\omega)$の関数$f\in\mathcal{A}(M)$で，$f$のHamilton方程式が定数以外に周期解をもたないならば　$c_0(M,\omega)=\infty$となります。　そのような$(M,\omega)$の例として4次元torus $T^4$上の次のsymplectic formを考えます；

$$\omega=dp_1\wedge dq_1+dp_2\wedge dq_2+b\,dq_1\wedge dq_2$$

これは任意の定数$b\in\mathbb{R}$に対し非退化で$T^4$上のsymplectic formとなります。$f\in\mathcal{A}(T^4)$として

$$f(p,q)=\cos(2\pi p_1)$$

をとると，$f$のHamilton vector場$X_f$は

$$X_f=-2\pi\sin 2\pi p_1\left(b\frac{\partial}{\partial p_2}-\frac{\partial}{\partial q_2}\right)$$

となり，$b$を無理数にとれば　$\dot{x}=X_f(x)$は周期解は定数以外には存在せず$f\in\mathcal{A}_1(M,\omega)$

となり 従って 任意の定数 $a$ に対し $a\cdot f \in A_1$ となるので $c_0(T^4, \omega) = \infty$ です。

周期解の存在について；

逆にもし $c_0(M, \omega) < \infty$ が何らかの方法で示せたとすれば、 $\forall f \in A(M)$ (non const) の1つの regular value $l$ に対し どんな小さな $\varepsilon > 0$ をとっても $f^{-1}([l-\varepsilon, l+\varepsilon])$ 上には少なくとも1つの定数でない、$\dot{x} = X_f(x)$ の、周期解が存在することになります。すなわち 定数でない周期解が存在する level が $f(M)$ 上 dense に分布していることになります。

次に $M$ の超曲面 $S$ 上の閉特性曲線の存在について考えます。超曲面 $S$ の各接

空間 $T_xS$ で $\omega(X,Y)=0\ (\forall Y\in T_xS)$ となる方向 $X$ が1意的に定まりますが，この $X$-方向を $S$ 全体で考えた直線束を $S$ の特性直線束とよび $\mathcal{X}_S$ で表わします。また $\mathcal{X}_S$ の積分曲線を $S$ の特性曲線とよびます。

ところで，超曲面 $S$ ~~が~~の近傍で定ぎされた vector 場 $X$ で $L_X\omega=\omega$ かつ $X$ は $S$ に横断的となるものが存在するとき $S$ は contact type であるといいます。このような vector 場 $X$ の存在と次をみたす $S$ 上の 1-form の存在とは同値です； $d\eta=\omega|_S$ かつ $\eta(X)\neq 0$ if $X\in\mathcal{X}_S-\{0\}$。

今 $S$ は contact type の超曲面 in $M$ とし，$X$ は上の条件をみたす $S$ の近傍で定ぎ

された vector 場で，$X$ が生成する flow を $\exp(tX)$ と書くとき，$\exp(tX)^*\omega = e^t\cdot\omega$ ですから $(\exp tX)_* \mathcal{X}_S = \mathcal{X}_{(\exp tX)(S)}$ となります。従って $x : S^1 \to S$ が閉特性曲線なら $(\exp tX)\circ x$ は $(\exp tX)(S)$ 上の特性曲線です。$M$ 上の関数 $f \in \mathcal{A}(M)$ で $S$ の近傍では各 $\exp tX(S) \equiv S_t$ が $f$ の level surface となるようにとれば（もっと強く，$f(\exp tX(S)) = t + \text{const}$ ），$f$ の Hamilton vector 場 $X_f$ を $S_t$ に制限したものは $\mathcal{X}_{S_t}$ の section です。従って $f$ の Hamilton 方程式の周期解と $S_t$ の閉特性曲線とは一致します。これらのことから $S$ が contact type で compact な超曲面であってもし $c_0(M,\omega) < \infty$ であれば $S$ 上に少なくとも

1つの閉特性曲線が存在します。

$\mathbb{R}^{2n}$ の star-shaped な領域をかこむ hypersurface は contact type です。実際 $0 \in \mathbb{R}^{2n}$ に関して star-shaped なら vector 場 $X$ として $X(p,q) = \frac{1}{2}\sum\left(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}\right)$ をとればよいからです。一方 $B^{2n}(1)$ の，従って任意の有界領域 $D$ の capacity $c_0(D)$ は有限であることがわかっているので，以上のことから $\mathbb{R}^{2n}$ の閉じた star-shaped な領域をかこむ超曲面上には少なくとも1つの閉特性曲線が存在することがわかりました。

Capacity とくみ出しエネルギー：

$(\mathbb{R}^{2n}, \omega_0)$ からそれ自身への symplectic 微分同相で support が cpt なもの全体のなす群を $\mathcal{D}$ で表わします。滑らかな path $\gamma : [0,1] \to \mathcal{D}$ に対し，$L_{\dot\gamma(t)}\omega_0 = 0$ なので（$L$ は Lie 微分）関数 $f_t \in C_c^\infty(\mathbb{R}^{2n})$ で $-df_t = i_{\dot\gamma(t)}\omega_0$ となるものが一意的に定まります。$H(t,x) := f_t(x)$ とおけば $H \in C_c^\infty([0,1] \times \mathbb{R}^{2n})$ が決まります（$C_c^\infty$ の右下の $c$ は support compact の $c$ です）。逆に $H \in C_c^\infty([0,1] \times \mathbb{R}^{2n})$ を与えれば $H$ が生成する Hamilton 流を積分して $id_{\mathbb{R}^{2n}}$ を始点とする path $\varphi_H : [0,1] \to \mathcal{D}$ が得られます。今 $\mathcal{D}_0 := \{\varphi_H(1) \in \mathcal{D} \mid H \in C_c^\infty([0,1] \times \mathbb{R}^{2n})\}$ とおけば $\mathcal{D}_0$ は写像の合成に関して群に

なっていますが，具体的に，~~積~~積に対応する関数を書くことができます。すなわち $H\# K(t,x) := H(t,x) + K(t, \varphi_H^{-1}(t)(x))$ とおけば $\varphi_{H\# K}(1) = \varphi_H(1)\circ\varphi_K(1)$ となります。

$f \in C_c^\infty(\mathbb{R}^{2n})$ の norm $\|f\|$ を $\|f\| := \max(f) - \min(f)$ とおけば 上の path $\gamma = \varphi_H : [0,1] \to \mathcal{D}_0$ の長さが

$$\mathrm{length}(\gamma) := \int_0^1 \|H_t\|\, dt$$

として定義されます。ただし $H_t(x) = H(t,x)$。そこで 各 $\varphi \in \mathcal{D}_0$ の energy $E(\varphi)$ を

$$E(\varphi) := \inf\{\mathrm{length}(\gamma) \mid \gamma : [0,1] \to \mathcal{D}_0 : C^\infty,\ \gamma(0) = id,\ \gamma(1) = \varphi\}$$

と定義します。$E(\varphi^{-1}) = E(\varphi)$ が成り立つことから $\mathcal{D}_0$ 上の擬距離 $d$ を

$$d(\varphi,\psi) := E(\varphi^{-1}\circ\psi)$$

と定ぎします。これは両側不変になっています。これが距離になっていること，すなわち「$E(\varphi)=0 \Leftrightarrow \varphi=id$」はそれほど単純なことでは有りません。このことを示すためとそれから次に述べるくみ出しエネルギーと先の capacity $c_0$ との関係を調べるためにも $\varphi$ のある特別な固定点（すなわち，時間依存 Hamilton 方程式 $\dot{x}(t)=X_{H_t}(x(t))$ の周期解）の作用積分の値（それを $\gamma(\varphi)$ と書く）が役に立ちます。このあたりのからくりがよくわかると symplectic キカ がもう少しよく理解できると思うのですが…。

$\mathbb{R}^{2n}$ の open set $U$ のくみ出しエネルギー $e(U)$ を

$$e(U) := \sup_{\substack{K\subset U\\ K:\text{有界}}} \inf_{\substack{\varphi\in\mathcal{D}_0\\ \varphi(K)\cap K=\phi}} E(\varphi)$$

と定義します。くみ出しエネルギとHofer-Zehnderのcapacityとの間に次の不等式が成り立つことをHoferが示しました；

$$c_0(U) \leq e(U) 。$$

一方 $e(B^2(1) \times \mathbb{R}^{2n-2}) \leq \pi$ であることは簡単に示せます。(実際、$e(B^2(1)) \leq \pi$ を示せばよいのですが、$B^2(1)$ と正方形 $U = (0, \sqrt{\pi}) \times (0, \sqrt{\pi})$ とは symplectic 同型ですし、$\varphi(U) \cap U = \phi$ で $E(\varphi) < \pi + \varepsilon$ ($\varepsilon > 0$ 十分小) なる $\varphi$ は、~~有~~ $U$ の nbd で $H(p_1, q_1) = q_1$ となる関数 $H \in C_c^{\infty}(\mathbb{R}^2)$ を使って作れるので OK です。) くみ出しエネルギー $e$ が capacity の axiom A1 をみたすことは $E(\psi^{-1}\varphi\psi) = E(\varphi)$ ($\forall \psi \in \mathcal{D}$) より明らかで、上の不等式より

$$\begin{array}{ccccc} e_0(B^{2n}(1)) & \leq & e(B^2(1)\times\mathbb{R}^{2n-2}) & \leq & \pi \\ \vee\!\mid & & \vee\!\mid & & \\ \pi \leq c_0(B^{2n}(1)) & \leq & c_0(B^2(1)\times\mathbb{R}^{2n-2}) & & \end{array}$$

であることがわかりました（横の不等式は示すのが簡単でタテが難しい）。従って $e$ が symplectic capacity であることと同時に $c_0$ もそうであることがたしかめられます。

タテの不等式や擬距離 $d$ が距離であることを示すのに使う関数 $\gamma : \mathcal{D}_0 \to \mathbb{R}$ の話をするため少々こまかい説明をします。

$\mathbb{R}^{2n}$ 内の $C^\infty$-級 closed loop $C^\infty(S^1, \mathbb{R}^{2n})$ を完備化した空間

$$H^{1/2}(S^1, \mathbb{R}^{2n}) := \left\{ x(t) = \sum_{j\in\mathbb{Z}} e^{2\pi j t J} x_j \;\middle|\; \sum_{j\in\mathbb{Z}} |j|\,|x_j|^2 < \infty \right\}$$

を考えます、ただし $J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}$ です。$H^{1/2}(S^1, \mathbb{R}^{2n})$ の部分空間 $E^+, E^0, E^-$ を $j$ に関する和が $j>0$, $j=0$, $j<0$ だけのものとして定義します。

$H^{1/2}$ からそれ自身への homeom. $h$ で次の形に表されるものを考えます;

$$h(x)=e^{\rho_+(x)}x^+ + x^0 + e^{\rho_-(x)}x^- + k(x)$$

ただし $x=x^+ + x^0 + x^-$ は分解 $H^{1/2}=E^+\oplus E^0\oplus E^-$ に対応するもので、$\rho_\pm : H^{1/2}\to\mathbb{R}$, $k: E\to E$ は連続かつ $k$ は有界集合を precompact 集合に写し、更に $\cancel{\rho_\pm(x^+)=0}$ 十分 norm の大きい $x^+\in E^+$ に対しては $\rho_\pm(x^+)=0$, $k(x^+)=0$ となるものとします。($x\in H^{1/2}$ の norm $\|x\|$ は $\|x\|^2=|x_0|^2+\sum |j|\,|x_j|^2$。)

このような $h$ による $E^+$ の像 $F=h(E^+)$ の全体を $\mathcal{F}$ とおき、$H\in C_c^\infty([0,1]\times\mathbb{R}^{2n})$ に対し

$$\gamma(H) := \sup_{F \in \mathcal{F}_1} \inf_{x \in F} a_H(x)$$

とおきます。

もし 2つの関数 $H_1, H_2 \in C_c^\infty([0,1] \times \mathbb{R}^{2n})$ から定まる symplectic diffeom $\varphi_{H_1}(1)$, $\varphi_{H_2}(1)$ とが一致していれば $\gamma(H_1) = \gamma(H_2)$ となるのでこれを $\gamma(\varphi)$ $(\varphi = \varphi_{H_i}(1))$ と書きます；
$\gamma : \mathcal{D}_0 \to \mathbb{R}$。

この関数 $\gamma$ を使って次をしめすことができます。

Prop $\psi \in \mathcal{D}_0$ に対し $H \in C_c^\infty(\mathbb{R}^{2n})$ は次をみたすものとする；

$$\begin{cases} \psi(\operatorname{supp} H) \cap \operatorname{supp} H = \phi \\ \min H < -E(\psi) \end{cases}$$

このとき Hamilton 方程式 $\dot{x} = X_H(x)$ は周期が1以下の$\underbrace{\text{周期解}}_{\text{定数でない}}$をもつ。

$\gamma(\varphi_H)$ は汎関数 $a_H : H^{1/2}(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ の臨界値であることが重要な事実で，あとはPropの条件をみたす $\psi \in \mathcal{D}_0$，$H \in C_c^\infty(\mathbb{R}^{2n})$ に対し $\mathrm{Fix}(\psi) = \mathrm{Fix}(\psi \circ \varphi_H(1))$ が成り立つことに注意すれば $\gamma(\varphi_H(1)) \leq E(\psi)$ を示すことができ，また方程式 $\dot{x} = X_H(x)$ が周期が1以下の定値でない周期解をもたなければ $\mathrm{Fix}(\varphi_H(t)) = \{H の臨界点\}$ $(0 < t \leq 1)$ であることから $\gamma(H) = -\min H$ を示すことができ，結局Propが得られます。

このPropの系として

<u>Cor 1.</u> $\varphi \in \mathcal{D}_0$ $\varphi \neq id \Rightarrow E(\varphi) > 0$

<u>Cor 2.</u> $c_0(U) \leq e(U)$ $\forall U \subset \mathbb{R}^{2n}$ open

が得られます。実際もし $\varphi \neq id$ なら $\varphi(B^{2n}(x_0, \varepsilon)) \cap B^{2n}(x_0, \varepsilon) = \phi$ となるように $\varepsilon$-ball をとり，$H \in \mathcal{A}(B^{2n}(x_0, \varepsilon))$ として

関数 $\pi|x-x_0|^2$ を $\partial B^{2n}(x_0,\varepsilon)$ の近くで適当に dump させ 更に $H(\partial B^{2n}(x_0,\varepsilon))=0$ となるように const>0 をひけば $H\in\mathcal{A}_1(B^{2n}(x_0,\varepsilon))$ であって かつ $-\min H>0$ となるので結局 $E(\psi)\geq-\min H>0$ となります。 また Prop をみとめてしまえば $c_0$ と $e$ の定義より $c_0\leq e$ は明らかでしょう。

参考文献：


[1] Hofer - Zehnder : Symplectic Invariants and Hamilton Dynamics, Birkhauser 1994


この本には詳しい文献リストがありますのでそれを参照して下さい。

# ３階の微分方程式と接触多様体の接続

TWISTOR理論、幾何構造と微分方程式

の一部として

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## §0. Introduction

Twistor理論を構成するdouble fibrationのmodelとして，実non-compact groupの部分群による商空間を考え，それぞれの幾何学的意味とその間の関係を捕え，曲った空間に適用すると，非常に面白いものが，沢山あらわれて来る，それらには，self-duality，Einstein-Weyl空間の意味づけ，更には，今まで研究されていなかったにもかかわらず，興味深い幾何構造達がある。下では無限次の構造が，上に持ちあげることで，有限次の構造を探すことを容易にする。

このTwistorの理論は，実はE. Cartanのいくつもの論文の中で重要なkey-pointになっていて，そのように考えることにより彼の論旨を明快に解釈出来る。このmodel空間達はjet空間のmodel空間にもなっているので，微分方程式の研究の必然的な舞台となり，Lie環はそのmodel空間の自己同型環として表われて来る。

このgraded Lie algebraとconnectionはTanakaにより，すべて統一的な原理によって支配されていることが示され，幾何構造の間

の関係が，明白に捕えられるようになる。

semi-simple な real Lie group とその R-space からなる 4つの diagram を次頁に与えるが，それらはそれぞれ興味深い幾何構造を与えているが，graded Lie algebra の次数はいろいろである。

勿論，複素 Lie 群で考えることも出来て，それらは又別の面白さがある。

この他にも例外 Lie group $G_2$ に対する diagram も考えられ，Cartan の 5 変数の論文と関係する。

semi-simple でない，例えば $\mathbb{R}^n$ の affine 変換群を考えて，その商空間で diagram を考えるのも，面白いようである。

このノートでは 4つの group のうちの最後の $Sp(n, \mathbb{R})$ （の主に左側の fibration）に関するものを説明する。

そのおゝまかなあらすじは次の通り。

左下 $Sp(n.\mathbb{R})/H_1 = U(n)/U(n-1) = S^{2n-1}$ の isotropy group $H_1$ は linear 表現として $GL(2n-1)$ の linear contact subgroup $Cont(2n-1)$ に surjective に写され，(無限次の構造である) 接触構造を定めている。上の $Sp(n.\mathbb{R})/H_{12} = U(n)/SO(n-1)$ は $S^{2n-1}$ 上 fiber が $U(n-1)/SO(n-1)$ である) の Lagrangean－Grassmann 束に等しい。$S^{2n-1}$ を1回の jet 空間 $J^1(n-1, 1)$ の compact 化と考えると，$Sp(n.\mathbb{R})/H_{12}$ は $J^2(n-1.1)$ の compact 化に等しい，その isotropy group は $Sp(n.\mathbb{R})/H_{12}$ の接空間の nilpotent gradation の graded Automorphisms の真の部分群で，3次の偏微分方程式系を定めた時のその Automorphism group と一致する。その方程式系を与えることにより，有限次の G－構造となり，Tanaka の理論より，unique な $sp(n.\mathbb{R})$-valued Cartan 接続が定まる。

その偏微分方程式系の解全体のなす空間が右下に与えられた $Sp(n.\mathbb{R})/H_2 = U(n)/SO(n)$ である。その空間 $Sp(n.\mathbb{R})/H_2$ は一方

Lagrangean str (即ち tangent bdle の $n$ 次元ベクトル束の symmetric 2 product と同型) という構造を持っている。上の空間 $Sp(n.\mathbb{R})/H_{12}$ は $Sp(n.\mathbb{R})/H_2$ の Lagrangean str の null plane 全体のなる空間とも考えられる。

この model を曲った空間に適用すると次のようになる。接触多様体に対し、その上の Lagrangean-Grassmann 束を考え、更に微分方程式系(即ち、その tangent bdle の tautological distribution 束の垂直方向に対する水平方向の splitting)を与えると、それに対応する $sp(n.\mathbb{R})$-接続が構成される。これは Lagrangean-Grassman 束上の接続であるが、不変ならば、接触多様体上に定義され、接触多様体の partial connection が定義されたとしようというのが、主なidea である。

これは、Cartan が 2階の常微分方程式系を与えることが、多様体の接続(=射影構造)を定めることと定義したものと、全く同じで、

あり，無限次元の接触構造に3階の微分方程式系を与えることが，接触構造に接続を定めることと同じになる。

この接触構造の接続の第1特性類としては，自然に接触構造のホモトピー類があらわれ，第2特性類が異種の接触構造と関係を持つものと考えられる。

$SL(n+1,R)$

$$
\begin{array}{ccc}
& SO(n+1)/SO(n-1) & \\
\swarrow & & \searrow \\
SO(n+1)/SO(n) & & SO(n+1)/SO(n-1)\times SO(2) \\
\text{proj. str.} & & \text{Grass. str}
\end{array}
$$

$SO(n+1,2)$

$$
\begin{array}{ccc}
& SO(n+1)\times SO(2)/SO(n-1) & \\
\swarrow & & \searrow \\
SO(n+1)\times SO(2)/SO(n-1)\times SO(2) & & SO(n+1)\times SO(2)/SO(n) \\
\text{Lie contact str.} & & \text{Lorentz} = \text{(n,1)-str.}
\end{array}
$$

$SO(n,n)$

$$
\begin{array}{ccc}
& SO(n)\times SO(n)/SO(n-1) & \\
\swarrow & & \searrow \\
SO(n)\times SO(n)/SO(n) & & SO(n)\times SO(n)/SO(n-1)\times SO(n-1) \\
\text{Pure spinor str.} & & \text{neutral} = \text{(n,n)-str.}
\end{array}
$$

$Sp(n,R)$

$$
\begin{array}{ccc}
& U(n)/SO(n-1) & \\
\swarrow & & \searrow \\
U(n)/U(n-1) & & U(n)/SO(n) \\
\text{contact str.} & & \text{Lagrangean str.}
\end{array}
$$

## §1. Lie algebra, jet space, cohomology

Lie 環 $\mathfrak{sp}(n.\mathbb{R})$ を復習しよう

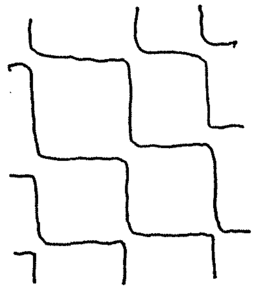
$$\mathfrak{sp}(n.\mathbb{R}) = \{ X \in \mathfrak{gl}(2n:\mathbb{R}) \mid {}^tXJ + JX = 0 \}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \right\}$$

但し $J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ 　$A'$は逆対角線に対する折り返し.

よって

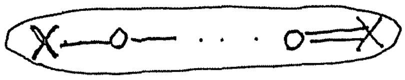
$$\begin{array}{c|c|c|c|c} & 1 & n-1 & n-1 & 1 \\ \hline 1 & a & {}^t d & {}^t e & j \\ n-1 & b & X & Z & e \\ n-1 & c & Y & -X' & -d \\ 1 & i & {}^t c & -{}^t b & -a \end{array}$$



と表わすことが出来て graded Lie algebra

$$\begin{array}{ccccccccccccccc} 2n^2+n & & 1 & & n-1 & & \frac{(n-1)(n+2)}{2} & & (n-1)^2+1 & & & & & & \\ \mathfrak{sp}(n.\mathbb{R}) & = & \mathfrak{g}_{-3} & \oplus & \mathfrak{g}_{-2} & \oplus & \mathfrak{g}_{-1} & \oplus & \mathfrak{g}_{0} & \oplus & \mathfrak{g}_{1} & \oplus & \mathfrak{g}_{2} & \oplus & \mathfrak{g}_{3} \\ & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & & \{i\} & & \{c\} & & \left\{\begin{matrix} b \\ Y \end{matrix}\right\} & & \left\{\begin{matrix} a \\ X \end{matrix}\right\} & & \left\{\begin{matrix} d \\ Z \end{matrix}\right\} & & \{e\} & & \{j\} \end{array}$$

を得る

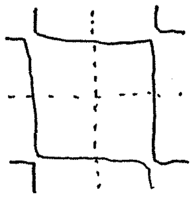


$\{\alpha_1, \alpha_n\}$

$$\boxed{Sp(n.\mathbb{R})/H_{12} = U(n)/SO(n-1)}$$

更に ambitwistor 対応 $\boxed{Sp(n,\mathbb{R})/H_1 = U(n)/U(n-1)}$, $\boxed{Sp(n,\mathbb{R})/H_2 = U(n)/SO(n)}$

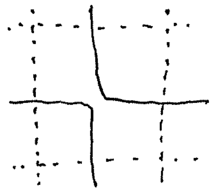
は 次の gradation が対応する



$$\mathfrak{sp}(n,\mathbb{R}) = \overset{1}{\mathfrak{g}_{-2}} \oplus \overset{2n-2}{\mathfrak{g}_{-1}} \oplus \overset{2n^2-3n+2}{\mathfrak{g}_0} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

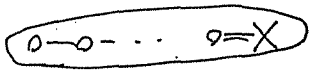
$$\begin{array}{ccccc} \Psi & \Psi & \Psi & \Psi & \Psi \\ \{i\} & \left\{\begin{array}{c} b \\ c \end{array}\right\} & \left\{\begin{array}{c} a \\ X \\ Y \\ Z \end{array}\right\} & \left\{\begin{array}{c} d \\ e \end{array}\right\} & \{f\} \end{array}$$

$\{\alpha_1\}$



$$\mathfrak{sp}(n,\mathbb{R}) = \overset{\frac{n(n+1)}{2}}{\mathfrak{g}_{-1}} \oplus \overset{n^2-1}{\mathfrak{g}_0} \oplus \mathfrak{g}_1$$

$$\begin{array}{ccc} \Psi & \Psi & \Psi \\ \left\{\begin{array}{c} i \\ c \\ Y \end{array}\right\} & \left\{\begin{array}{c} a \\ X \\ b \\ d \end{array}\right\} & \left\{\begin{array}{c} j \\ e \\ Z \end{array}\right\} \end{array}$$



$\{\alpha_n\}$

$H^{*,q}(\mathfrak{m}, \mathfrak{g})$ の non-vanishing part は 次の通りである ([Yamaguchi])

$\{p_{12}\} \in H^{2,2}$, $\{p_{21}\} \in H^{3,2}$ $\quad n=2$

$\{p_{12}\} \in H^{0,2}$, $\{p_{1n}\} \in H^{0,2}$ $\quad n \geq 3$

$\{p_{12}\} \in H^{2,2}$ $\quad n=2$

$\{p_{12}\} \in H^{1,2}$ $\quad n \geq 3$

$\{p_{21}\} \in H^{2,2}$ $\quad n=2$

$\{p_{n\,n-1}\} \in H^{0,2}$ $\quad n \geq 3$

$Sp(2\cdot\mathbb{R})$ の場合

$$\{P_{12}\} \in Y \otimes (i^* \wedge b^*)$$

$$\{P_{21}\} \in Z \otimes (c^* \wedge Y^*)$$

が確かめられる。

ここで大事な Observation は次の事実である。

Double fibration

$$\begin{tikzcd} & U(n)/SO(n-1) \arrow[dl] \arrow[dr] & \\ U(n)/U(n-1) & & U(n)/SO(n) \end{tikzcd}$$

CONTACT STR. LAGRANGEAN STR

において contact mfd $U(n)/U(n-1) = S^{2n-1}$

を $J^1(n-1,1) = \{1\text{-jet of } \mathbb{R}^{n-1} \to \mathbb{R}\}$ の model space

とした時

$$\boxed{U(n)/SO(n-1) = J^2(n-1,1)} = \{2\text{-jets of } \mathbb{R}^{n-1} \to \mathbb{R}\}$$

一方 Lagrangean mfd $U(n)/SO(n)$ において

$$U(n)/SO(n) = \{1\text{-dim of Lag. space}\}$$

と自然に見ることが出来る。

これは
$$\begin{array}{ccc} U(n)/SO(n-1) & \longleftarrow & U(n-1)/SO(n-1) \\ \downarrow & & \\ U(n)/U(n-1) & & \end{array}$$

という fibration により $U(n)/SO(n-1)$ は contact manifold $U(n)/U(n-1) = S^{2n-1}$ の Lagrangean subspace 全体のなす空間と見なされるが、これが $J^2(n-1, 1)$ に等しいということは古典的な結果（Bäcklund, Yamaguchi）である。

$n=2$ の時 Lie 環の同型 $sp(2.\mathbb{R}) \cong so(3.2)$ より Lagrangean structure は (2.1)-conformal structure と等しくなり

$$U(2)/SO(1) = S^1 \times S^3 = SO(3.2)/H_{12}$$

は (2.1)-metric を持つ conformal flat 空間 $S^2 \times S^1$ の null vector 全体に等しくなり、null vector を表わす式は
"equation of Monge of the second order"
(Chern 1940) となる。

## §2. 3階の偏微分方程式系

$n-1$ コの変数 $x_1, \ldots, x_{n-1}$ 上の関数

$y = y(x_1, \ldots, x_{n-1})$ を考える

$$J^1(n-1, 1) = \{x_1, \cdots, x_{n-1}, y, y'_1, \ldots, y'_{n-1}\} \qquad y'_i = \frac{\partial y}{\partial x_i}$$

$$J^2(n-1, 1) = \{x_1, \ldots, x_{n-1}, y, y'_1, \ldots, y'_{n-1}, y''_{ij}\}$$

$$y''_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j} \qquad (1 \leq i \leq j \leq n-1)$$

3階の偏微分方程式系を

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = F_{ijk}(x, y_1, \ldots y_{n-1}, y'_1, \ldots, y'_{n-1}, y''_{ij})$$

$$(1 \leq i \leq j \leq k \leq n-1)$$

で与える。

これを $J^1(n-1, 1)$ の contact 変換 の $J^2(n-1, 1)$ への引き上げであって、解 ($(n-1)$次元平面) を解に移すという同値で同値問題を考えよう。

3階の偏微分方程式系を $(x_1, \ldots x_{n-1}, y)$ という $n$ 次元空間の微分同相による同値で同値問題を考えるといっても同じことである。

$\dfrac{(n-1)(n+2)}{2} + n-1+1 = \dfrac{n^2+3n-2}{2}$ 次元の多様体.

$J^2(n-1, 1)$ 上の $\mathbb{R}^{\frac{n^2+3n-2}{2}}$ valued 1-form $\omega$ $= {}^t(\omega_1, \ldots\ldots, \omega_{\frac{n^2+3n-2}{2}})$ を次のように与える

$$\omega_1 = dy - \sum y'_i\, dx_i \qquad 1\text{コ}$$

$$\left.\begin{aligned}
\omega_2 &= dy'_1 - \sum y''_{1i}\, dx_i \\
&\vdots \\
\omega_n &= dy'_{n-1} - \sum y''_{n-1\,i}\, dx_i
\end{aligned}\right\} (n-1)\text{コ}$$

$$\left.\begin{aligned}
\omega_{n+1} &= dy''_{11} - \sum F_{11i}\, dx_i \\
&\cdots \\
\omega_{\frac{n^2+n}{2}} &= dy''_{nn} - \sum F_{nni}\, dx_i
\end{aligned}\right\} \frac{n(n-1)}{2}\text{コ}$$

$$\left.\begin{aligned}
\omega_{\frac{n^2+n}{2}+1} &= dx_1 \\
&\cdots\cdots\cdots \\
\omega_{\frac{n^2+3n-2}{2}} &= dx_{n-1}
\end{aligned}\right\} (n-1)\text{コ}$$

この coframe において. 接触変換で解を解に移す 1次変換は

$$G = \begin{array}{c} \begin{array}{cccc} 1 & n-1 & \frac{n(n-1)}{2} & n-1 \end{array} \\ \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & 0 & * \end{pmatrix} \end{array} \begin{array}{l} 1 \\ n-1 \\ \frac{n(n-1)}{2} \\ n-1 \end{array} \subset GL\left(\frac{n^2+3n-2}{2}, \mathbb{R}\right)$$

で与えられる

同値問題は 2つの3階偏微分方程式系, $\omega, \tilde{\omega}$ が与えられた時, $\varphi : J^2(n-1, 1) \longrightarrow J^2(n-1, 1)$ local diffeo で

$$\varphi^* \omega = g \tilde{\omega}$$

となるような $J^2(n-1, 1)$ 上の $G$-valued function の存在するものをみつけることが出来るかという問題となる。

$\omega \in \Gamma(F^* U)$ とみなし. $\omega G \subset F^* U$ という $G$-subbundle of the frame bundle ($U \subset J^2(n-1, 1)$) を考える。

ここで、最も基本的で、あまり教科書には明言されていない大事な定理を思い出そう。

<u>定理</u> coframe bundle の subbdle の bdle isomorphism が、その base mfd に induce する diffeo を微分したものに等しい条件は canonical form を保つことである。

ここで coframe bdle の canonical form とは $\mathbb{R}^n$-valued 1-form $\theta$ on $F^* M$ で

$$\theta(X)_a = a^{-1} \pi_*(X) \quad \{a : \mathbb{R}^n \to T^* M\} \in F^* M$$

と定義されるもの。

coframe bundle の subbdle $\omega G$ に
canonical form $\theta$ を制限したものは
まさに $\quad g\omega \quad$ at $(\omega\cdot g) \in \omega G$

であるから，$\theta$ を考えても $g\omega$ を考えても
同じことで，classical な Cartan，Chern は
$\omega$ を微分するのだが，Sternberg の教科書
では，canonical form を考えて微分するという
明確化を与えている。

このV上での $\mathbb{R}^n$-valued form と subbundle
上の canonical form と呼ばれる $\mathbb{R}^n$-valued
form の混在が Cartan の方法の極意
のひとつで，わかりにくいところでもあり，
Singer–Sternberg §2.14, Lexicon to Cartan
で説明が与えられている。

## §3. Reduction, Prolongation と接続の構成

同値問題の設定が出来たので、この問題を解こう。実は $n=2$ の場合には Chern が行っていて、ある意味では答を出している。その方法は、古典的な方法なので、まず、$G$ の構造方程式を考えて、その構造関数を考えるのであるが、connection を変える（＝horizontal 方向を変える）という方法で、構造関数をコホモロジーの元として簡単なものに出来ることを示し、従ってそれ故、群 $G$ が、その構造関数（定数関数となる）の isotropy 群に reduce するという Reduction の process を何度か行う。

その reduction の最後に構造関数が出てくるので、それを $I$ と書き、（$F$ の $x, y, y', y''$ の3階までの微分迄で表わされる）、それが消えている時、isotropy 群がより小さくならず、もう reduction は出来ないので、prolongation を行う。このようにして最終的には $sp(2, \mathbb{R})$-valued 1-form $\Theta$ が unique に定まり、その不変量として5つの関数 $a, b, c, d, f$ が出る。それが消えていれば

$y'''=0$ と同値というのが最初の結論。

ここで $I=0$ という条件は、次のような意味を持つことも示している。すなわち、

$J^2(1,1)$ 上の3階微分方程式で定まる解曲線が、Twistor Picture で示した、解曲線全体のなす Lagrangean str（この場合 (2,1)-共形構造）の2階の Monge equation で表わされる null-curve と等しい。

これは Wünschmann の結果であるが、Chern は、Cartan connection ⓐ が (2,1)-metric の conformal connection の引き戻しに等しいということで証明を与えている。

更に $I \neq 0$ という場合には、群を更に reduce することが出来、このあと prolongation を行い、最終的に non-semi-simple な6次元の群の Lie 環に値を持つ Cartan connection を unique に構成している。この場合にも沢山の invariant が出て来て、それが消えていれば

$$y'''-y=0$$

という方程式と同値となるというのが次の結論。
（6次元の群は上の方程式の同型群）

## Chernの仕事の問題点

勿論、Chernの仕事は巧妙で美しいが、いくつかの問題点が残っている。

ひとつめは、$I=0$ あるいは $I\neq 0$ の場合の出てくる不変量関数の数の多さで、スペンサーコホモロジーの計算によると、独立なものは例えば $I=0$ の場合は、1コのみで、残りは、その微分等より出てくるはずである。Flatになる為にはその1コさえ消えていればよい。その不変量関数も $F$ より具体的に求まるはずである。その不変量の幾何学的意味は、$J^1(n+1,1)$ のconnectionの引き戻しになっているかどうかを表わすもののはずである。

もうひとつの問題は、場合によって群が変化していることで、それはそれなりに意味があるが、我々は、$I\neq 0$ の場合も、それがひとつの曲率であって、常にいくつかの曲率が、最初の $F$ から定まる形で書けていて、そのvanishingの程度で、同値を決定づけるということが、より望ましい。

実は、これらの問題を一挙に解決しているのが<u>Tanakaの理論</u>である。(実際の計算は必要であるが).

すなわち、Tanaka は、$G^\# \subset GL(n)$ に対して、等質空間 $G/G'$ で $G'$ の isotropy representation の像が $G^\#$ になっているものを先に見つけておく。

この isotropy 表現は、$T_o(G/G')$ というベクトル空間に nilpotent graded Lie algebra を入れて、その automorphism となっているとする。

更に、$G$ は $G^\#$ の prolongation と等しいとする

この時に自然に $U$ 上の $G'$-principal bdle $P$ と $\mathfrak{g}$-valued Cartan connection が一意的に存在し、$U$ 上の $G^\#$-構造の同値問題がこの Cartan connection の同値問題に還元されるというのが結論である。Cartan connection から curvature が定義され、invariant となる。

この構成で更に注意すべきことは、Cartan connection の条件が

$$K^p = 0 \ (p<0), \quad \partial^* K^p = 0 \ (p \geq 0)$$

ということで，普遍性があり，Twistor の Double fibration に対してのcompatibility が存在する。これにより種々の構造が非常に見易くなり，最初にあげた4つの型の Double fibration に対して，それぞれ，面白い結果を得ることが出来る。

曲率が存在するのは，Lie 環から定まるスペンサーのコホモロジーが消えていない部分で，それは Kostant の定理により，Weyl 群のルートへの作用を調べることにより，計算が可能となる。

## §4. 接触多様体と3階偏微分方程式系

今，$M$ を $2n-1$ 次元接触多様体とする。その上の Lagrangean subspace 全体のなす空間 $L(M)$ は $(2n-1)+\dim \frac{U(n-1)}{SO(n-1)} = (2n-1)+\frac{n(n-1)}{2} = \frac{n^2+3n-2}{2}$ 次元多様体である．その上には local に

先程の $\omega_1=\omega_2=\cdots=\omega_n=0$ という2次の接触形式で定義される global な $\frac{n(n-1)}{2}+(n-1)=\frac{n^2+n-2}{2}$ 次元 $\overset{\text{contact}}{\text{distribution}}$ $D(L(M))$ が存在する。nilpotent な graded Lie alg $\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ に対応する $\mathfrak{g}_{-1}$-part と言ってもよく，又，$l\in L(M)$ において $\pi_*(X)=l$ となる $X\in L(M)$ ということも出来る．$D(L(M))$ は自然に projection $\pi: L(M)\longrightarrow M$ の vertical 成分すなわち fiber $U(n-1)/SO(n-1)$ に接する部分 $V$ を部分空間として含んでいる．

$L(M)$ は $M$ の Lagrangean Grassmann 束と呼ばれることもある。

Tanaka の理論を $L(M)$ の上の $G$-構造の Cartan 構造の接続の構成に適用する為には、$L(M)$ の接空間の gradation

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

をより細かくして、その graded Lie algebra isomorphism が isotropy 群と等しくなるようにしなければならない。即ち

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \overset{\frac{n(n-1)}{2}}{\mathfrak{g}_{-1}^{+}} + \overset{n-1}{\mathfrak{g}_{-1}^{-}}$$

と分解することによって始めて、

$$Sp(n.\mathbb{R})/H_{12} = U(n)/SO(n-1) = L(S^{2n-1})$$

の isotropy 表現の像

$$\rho : H_{12} \longrightarrow GL\left(\frac{n^2+3n-2}{2}\right)$$

と graded Lie alg. iso が等しくなるのである。

ちなみに $d\rho$ の像 $d\rho(\mathfrak{h}_{12}) \subset \mathfrak{gl}\left(\frac{n^2+3n-2}{2}\right)$ は

$$\begin{array}{cc} \begin{array}{cccc} 1 & n-1 & \frac{n(n-1)}{2} & n-1 \end{array} & \\ \begin{pmatrix} * & & & \\ * & * & & \\ * & * & * & \\ * & * & 0 & * \end{pmatrix} & \begin{array}{c} 1 \\ n-1 \\ \frac{n(n-1)}{2} \\ n-1 \end{array} \end{array}$$

という形であった。

以上の考察により、次の定義は至極、妥当なものと言えよう。

<u>定義</u> 接触多様体 $M^{2n-1}$ の Lagrangean-Grassmann 束 $L(M)$ の partial connection (partial spray) とは $l \in L(M)$ に対して 接触 distribution $D_l(L(M))$ の subspace $H_l$ のなめらかな対応で、

$$D_l(L(M)) = V_l \oplus H_l$$

を満たしているもの。

このようにして定義された水平空間 $H_l$ は $(n-1)$ 次元部分空間で、

$$\pi_*(H_l) = l \subset T_{\pi(l)}M$$

を満たしている。これは先程の言葉でいえば、<u>3階の偏微分方程式系</u>を与えることと言っても良いだろう（但し変数は $n-1$ 次元）。

我々は次の定理を得る.

> **定理** 接触多様体 $M^{2n-1}$ の Lagrangean-Grassmann 束 $L(M)$ に partial connection が与えられた時 $L(M)$ 上 $H_{12}$-principal bundle $P$ とその上の $sp(n.\mathbb{R})$-valued Cartan connection が一意的に構成される。

ここで $H_{12} \subset Sp(n.\mathbb{R})$ は $Sp(n.\mathbb{R})/H_{12} = U(n)/SO(n-1)$

$= L(S^{2n-1})$ となる non-compact group.

この normal connection の curvature は §1 に述べた Spencer cohomology の結果から

$n=2$ の時 $$H^{2.2},\ H^{3.2}$$

$n \geq 3$ の時 $$H^{0.2},\ H^{0.2}$$

に存在する。 $n=2$ の時の $H^{2.2}$-part の curvature が Chern の不変量 $I$ であり. double fibration による cohomology の移り方を見ることで, $I=0$ がまさしく connection が Lagrangean str $=$ (2.1)-conformal structure の引き戻しになっていることと一致し、Wünschmann の結果を得る。Chern は群を変えたので定義していないが、$H^{3,2}$-part の curvature $J$ が重要になろう。$J=0$ が connection が contact structure の引き戻しになっている為の条件であることを示すことが出来る。この $J$ を $y'''=F(x,y,y',y'')$ を定める $F$ の微分で表わすことも、面倒な計算であろうが、出来るであろう。

$n \geq 3$ 以上の時も同様に $I$, $J$ という2つの curvature を得ることができて、$J=0$ が connection が contact structure の上のものの引き戻しになっている為の条件であることが示せる。いいかえれば、$J=0$ という条件が、$L(M)$ 上の partial connection が $M$ 上の partial connection に落ちる為の条件であるということである。このような $L(M)$ 上の partial connection は "invariant" partial connection と equivalent であるとも言えよう。

ここで invariant partial connection on $L(M)$ = partial connection on $M$ の定義を与えよう。

最初に挙げた Double fibration の 4番目の左側を考える。

$$\begin{array}{ccc} Sp(n.\mathbb{R})/H_{12} = U(n)/SO(n-1) & \longleftarrow & H_1/H_{12} = U(n-1)/SO(n-1) \\ \swarrow & & \\ Sp(n.\mathbb{R})/H_1 = U(n)/U(n-1) = S^{2n-1} & & \end{array}$$

$Sp(n.\mathbb{R})$ の $S^{2n-1}$ への action の isotropy group $H_1$ は Lagrangean space $U(n-1)/SO(n-1)$ に transitive に作用している.

$Cont(2n-1) \subset GL(2n-1)$ を $((2n^2-n)$ 次元の$)$ 接触変換の1次部分のなす Lie gp. とする

$$Cont(2n-1) = \left\{ \begin{pmatrix} k & 0 \\ \xi & A \end{pmatrix} \middle| \ {}^tAJA = kJ \right\}$$

その時 $Cont(2n-1)/C_1 = U(n-1)/SO(n-1)$ で

$H_1$ の $U(n-1)/SO(n-1)$ への action は natural map

$\rho : H_1 \longrightarrow Cont(2n-1) \to 1$ を factor する.

$M^{2n-1}$ を contact manifold とすると，その contact str を保つ frame 全体 $F_{cont}(M)$ は $M$ 上の $Cont(2n-1)$-bundle となり，Lagrangean Grassman bdle $L(M)$ は $F_{cont}(M)$ に associate された fiber bundle とみることが出来る

$$L(M) = F_{cont}(M) \times_{Cont(2n-1)} Cont(2n-1)/C_1$$

$$\left( Cont(2n-1)/C_1 = U(n-1)/SO(n-1) \right).$$

<u>定義</u> 接触多様体 $M^{2n-1}$ の partial connection とは $L(M^{2n-1})$ の <u>invariant</u> partial connection 即ち．$\ell \in L(M)$ に対して接触 distribution $D_\ell(L(M))$ の subspace $H_\ell$ のきめらある対応で

1) $D_\ell(L(M)) = V_\ell \oplus H_\ell$

2) $H_{\ell a} = (R_a)_* H_\ell$ for ${}^\forall a \in Cont(2n-1)$

$n=2$ の時、接触多様体 $M^3$ の partial connection は covariant differentiation

$$\nabla_X Y \qquad X, Y \text{ は Lagrangean vector field}$$

を定めることが、geodesic flow が connection を定めることと同様の議論で示される。

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# On the Classification of Tight Contact Structures on the 3-Torus

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## §.0 Intro.

Contact structure は主に幾何の対象として古くから研究されてきました。最近になって、3次元 contact structure の微分トポロジー的研究が Eliashberg や Giroux 達によって行われ、色々と面白い性質がわかってきました。

さて、contact structure の定義を一言でいえば、"completely non-integrable" な超平面場のことです。Darboux の定理によればすべての contact structure は local には同型です。ですから、問題となるのは、その global な性質ということになります。

以下、本稿では、3次元の場合のみを考えます。

3次元の contact structure は大きく分けると2つの class があり、それぞれ、tight, over-twisted と呼びます。

over-twisted の分類は Eliashberg によってなされました。

それによると，その分類は 2-plane field としての homotopy class の分類と完全に一致します。従って，幾何学的に見て，面白い対象とはいえません。

一方，tight の場合ですが，tight な contact structure はある種の"rigidity"をもっていて，多様体の位相による強く制約を受けます。例えば，$S^3$ に入る tight contact structure は up to isotopy 及び up to orientation で unique です。$S^3$ の 2-plane field の homotopy class が可算無限個あることを考えると，非常に興味深い性質だと言えます。

今までのところ，分類の完成しているのは，$S^3$，$S^2 \times S^1$，$T^3$ ぐらいしかないようです。今後の進展が期待されるところです。

Remark　3次元 contact structure の概観については Ref. の 3 及び 5 を参照されるのがよいと思います。

## §1. Tight Contact Structures

Def. 1.1 $M^3$上の2-plane field $\xi$がcontact

$\iff$ $^\exists \alpha \in \Omega^1(M)$ s.t.

$\alpha$: non singular, $\alpha|_\xi = 0$

かつ $d\alpha|_\xi$: non degenerate

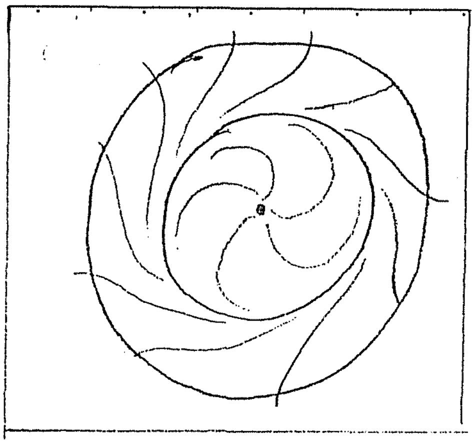
Remark $\alpha \wedge d\alpha$はvolume formであり、$\alpha$の代りに$-\alpha$を取っても、その符号は変わらない。よって$\xi$は$M$に向きを決める。

今、surface $F$が contact 3-mfd $(M, \xi)$に埋めこまれた状況を考える。このとき$TF \cap \xi$は、$\xi$が$F$に接する点を除いて、line fieldを定める。これを積分して得られた$F$上の特異codimension 1 foliationを、characteristic foliationと呼び、記号$ch_\xi(F)$で表す。

Def 1.2 $(M^3, \xi)$がtight

$\iff$ すべてのembedded disc $D \hookrightarrow M$について

$ch_\xi(D)$は closed leaf を持たない。



左図のような chr. fol. を持つ disc を over-twisted disc と呼ぶ。tight ならば定義より ov.-tw. disc を持たない。

tight な contact str. の例をいくつか挙げておく。

<u>Example</u> 1.3 $(\mathbb{R}^3,\ dz + x\,dy = 0)$

これを $\mathbb{R}^3$ の standard contact str. と言う。

<u>Example</u> 1.4 今 $S^3$: unit sphere $\hookrightarrow \mathbb{C}^2$ を考える。このとき $\xi := TS^3 \cap J(TS^3)$ で決まる 2-plane field $\xi$ は contact である。これを $S^3$ の standard contact str. と言う。$(S^3, \xi)$ の tight 性は Bennequin により示された。

<u>Remark</u> $(S^3 - \{1点\},\ \xi|_{S^3-\{1点\}}) \underset{\text{contact 同型}}{\cong} (\mathbb{R}^3,\ dz + x\,dy = 0)$

## §2 Main Results

以下 3次元 torus $T^3$ と $\mathbb{R}^3$ を integral lattice で割った空間を同一視する。

自然数 $n$ に対して，$T^3$ 上の 1-form $\alpha_n$ を

$\alpha_n := (\cos 2\pi nz)dx + (\sin 2\pi nz)dy$

で定義する。$\alpha_n$ の決める contact str. を $\xi_n$ と書く。

<u>Main Thm</u>

① $(T^3, \xi)$ を orientable tight contact str. とする。このとき $^{\exists}n \in \mathbb{N}$ 及び $^{\exists}f \in \mathrm{Diff}(T^3)$ があって，$f : (T^3, \xi) \longrightarrow (T^3, \xi_n)$ は contact 同型。

② $f : (T^3, \xi_n) \longrightarrow (T^3, \xi_m)$ が contact 同型ならば，$n = m$ かつ $f^*([dz]) = [dz]$。

①については §4 で，②については §3 で述べる。

## §3 Invariants of Tight Contact Str. on $T^3$

<u>Def</u>.3.1 $\ell$ : simple closed curve $\hookrightarrow T^3$ が linear である。

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}(a,b,c) \in \mathbb{Q}^3 - \{(0,0,0)\}$ s.t.

$\ell$は直線 $\{(at, bt, ct) \mid t \in \mathbb{R}\}$ を latticeで割ってできたcurveとisotopic。

linear curveの法bundleにはcanonicalに自明化が入る。つまり $\ell$ を含むincompressible torus $F$ を与えたとき、$TF|_\ell / T\ell$ は法bundleのsubbundleであるが、これによって決まる自明化のことである。

Def 3.2　$\ell$: linear Legendrian curve $\hookrightarrow (T^3, \xi)$ とする。このとき twisting number $tw_\xi(\ell)$ を

$$-tw_\xi(\ell) = \left\{\begin{matrix}\text{canonicalな自明化に対する} \\ \text{subbundle } \xi|_\ell / T\ell \text{ の degree}\end{matrix}\right\}$$

Thm 3.3　$tw_{\xi_n}(\ell) \geq n \cdot \left| \langle [\ell], [dz] \rangle \right|$

ただし $\langle\ ,\ \rangle$ は Kronecker 積。

又、この評価は各linear curveのisotopy classに対してbest possibleである。

Main Thmの②はこの定理の容易な系である。

Thm 3.3の証明の概略)

$l_0 := \{(0,0,t) \mid t\in\mathbb{R}/\mathbb{Z}\} \subset T^3$ とおく。

簡単のため $l$ が $l_0$ に isotopic な場合のみ示す。

Covering $\pi : \mathbb{R}^2\times S^1 \longrightarrow T^3$ を考える。

$$\begin{array}{ccc} \mathbb{R}^2\times S^1 & \longrightarrow & T^3 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x,y)\times(z) & \longrightarrow & (x,y,z) \end{array}$$

$\pi$ による $\xi_n$ の lift を $\tilde{\xi}_n$, $l$ の lift の1つを $\tilde{l}$ とおく.

一方 Example 2.4 で与えた $(S^3,\xi)$ において, 原点を通る Lagrangian 平面 $L$ をひとつとり, $k := S^3(1)\cap L$ とおくと, $k$ は $S^3$ における位相的に自明な Legendrian knot であり, しかも $tb(k) = -1$ となっている。

すると contact 同型 $\Psi : (\mathbb{R}^2\times S^1, \tilde{\xi}_n) \longrightarrow (TN(k), \xi|_{TN(k)})$ であって, $\Psi(l_0) = k$ であるようなものがつくれる。ここで $TN(k)$ は $k$ のある管状近傍である。(注: $\Psi$ は $(S^3,\xi)$ へのコンタクト埋めこみと思えるが, これより $\tilde{\xi}_n$ の tight 従って $\xi_n$ の tight 性が出る。)

$tw_{\xi_n}(l) = n-1$ として矛盾を出す。このとき, $\Psi(l)$ は $S^3$ の中で, $\Psi(l)$ を limit cycle にもつような embedded disc をはるが, これは $(S^3,\xi)$ の tight 性に反す。 □

## §4. Proof of Main Thm ①

証明のための道具は2つある。1つは Eliashberg による次の定理である。

<u>Thm 4.1</u>　$B^3$ の tight contact str. は up to ori. 及び up to isotopy で，与えられた $\partial B^3$ 上の chr. fol. に対して一意である。

この定理から思いつくのは，$T^3$ を適当な incompressible torus によって分割し，$B^3$ の case に帰着させることである。そのためには，まず chr. fol. の様子を詳しく見る必要がある。
以下，Giroux により開発された convex surface の理論について紹介しよう。

<u>Def 4.2</u> (Giroux)　$F$ : surface $\hookrightarrow (M^3, \xi)$ とする。
$F$ が convex $\iff$ $\exists X$ : contact flow s.t. $X \pitchfork F$

<u>Thm 4.3</u> (Giroux)　$F$ closed ori. surface $\hookrightarrow (M, \xi)$
このとき $F$ が convex であることと次は同値；
1. $\exists \Gamma$ : simple closed curves の disjoint union $\subset F$
s.t. $ch_\xi(F) \pitchfork \Gamma$

2. $F$ の codim 0 submfds $F^+, F^-$ があり
$\Gamma = F^+ \cap F^- = \partial F^+ = -\partial F^-$ かつ $F = F^+ \cup F^-$

3. $^\exists Y$ : flow on $F$ s.t.

1) $Y$ の積分曲線と $ch_\xi(F)$ は一致する。

2) $\operatorname{div} Y \begin{cases} > 0 & \text{on } F^+ \\ < 0 & \text{on } F^- \end{cases}$

（注）$F^+$ を positive
$F^-$ を negative
component と呼ぶ。）

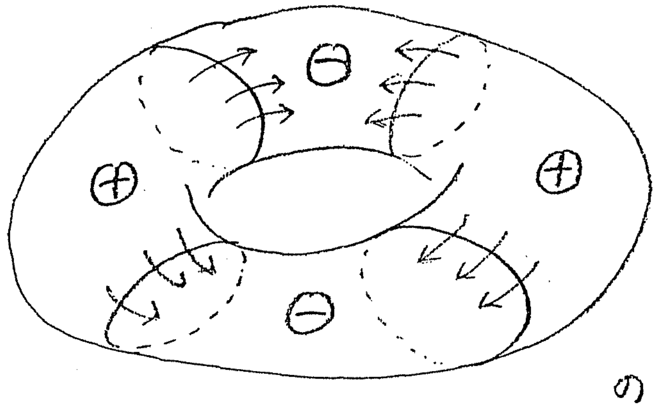
<u>Thm</u> 4.4 (Giroux) $F$ : convex surface $\hookrightarrow (M, \xi)$。
$\mathcal{F}$ : a singular foliation s.t. $\Gamma$ に対して
Thm 4.3 の 条件 1, 2, 3, を満たす。
$\Longrightarrow$ $F$ を isotopy により動かして $\mathcal{F}$ を chr. fol. として
実現できる。

以上から convex surface を調べるには $\Gamma$（分割集合と呼ぶ）の入り方にのみ注目すればよい。又, generic な closed. ori, emb. surface は convex であることがわかっている。

今 $(T^3, \xi)$ が tight としたとき, incomp. convex torus の chr. fol. はどうなっているだろうか？

実は次が成り立つ

<u>Prop.4.5</u> $T^2$: convex torus in tight contact manifold とする。このとき分割集合 $\Gamma_{T^2}$ は、偶数本の parallel な simple closed curve 達からなる。




左図は $\Gamma_{T^2}$ が4本の場合である。矢印は Thm 4.3 に出てきた flow $Y$ の流れを表している。

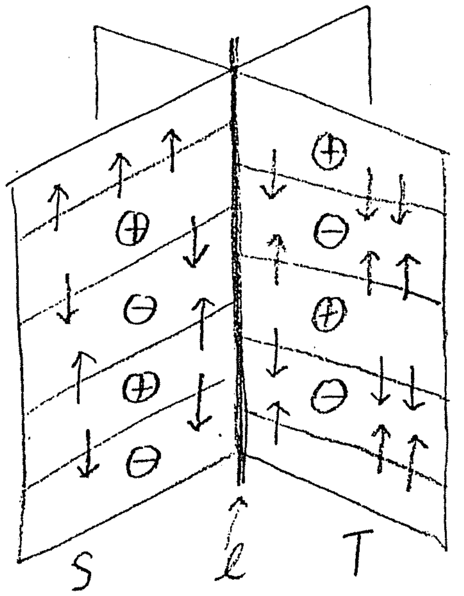
これに基づき、次を示すのが Main Thm ① の証明の一番の山である。

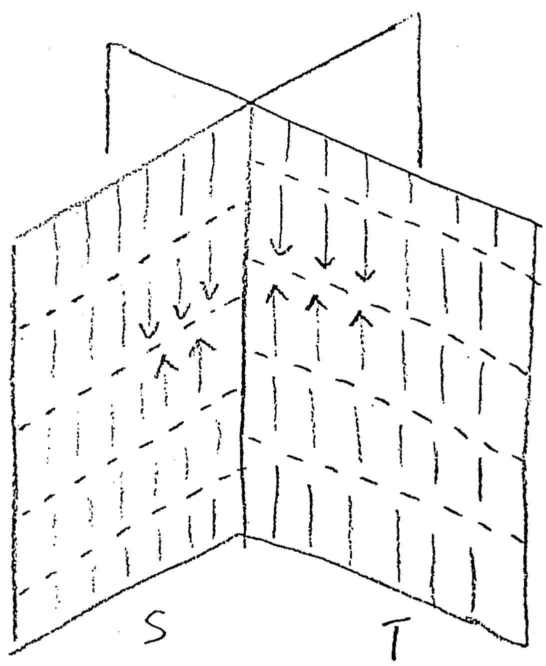
<u>Prop</u> 4.6 $(T^3, \xi)$ が tight とする。このとき次のような incomp. convex torus の pair $(S, T)$ がある。

1, $S \pitchfork T$ かつ $S \cap T$ は一本の Legendrian curve $l$ からなる。

2, $\# \Gamma_S = \# \Gamma_T = \# (l \cap \Gamma_S) = \# (l \cap \Gamma_T)$

3, $l$ は $\Gamma_S$ 及び $\Gamma_T$ と parallel でない。






上左図は $\#\Gamma_S = 4$ の場合である。Thm.4.4 を用いて
chr.fol が上右図になるようにする。
ただし点線で描かれている所は，chr.fol. の singular point の集合であり，$\Gamma_S, \Gamma_T$ と同本数で parallel な simple closed curve たちからなる。

さて，一般に emb. surface の周りの contact str. の germ は chr.fol. に対して unique だから，$S \cup T$ のある正則近傍 $N$ から $(T^3, \xi_n)$ への contact 埋めこみ $\psi$ であって，
$\psi(S) = \{(x, 0, z) \mid x, y \in \mathbb{R}/\mathbb{Z}\}$, $\psi(T) = \{(0, y, z) \mid y, z \in \mathbb{R}/\mathbb{Z}\}$
であるようなものがとれる。ただし $n = \frac{1}{2}\#\Gamma_S$ である。
$G := T^3 - N$ とおくと，これは solid torus であるが，
$N$ をうまくとると $\partial G = \partial \bar{N}$ は次をみたす。

1, $\partial G$ は convex

2, $\# \Gamma_{\partial G} = 2$

3, $\Gamma_{\partial G}$ の各 component は longitude をなす。

さて, $\psi$ を contact 同型 $\Phi : (T^3, \xi) \to (T^3, \xi_n)$ に拡張するには, $ch_\xi(\partial G)$ を fix したとき, $G$ に入りうる tight contact str. が一意であることを言えばよい。

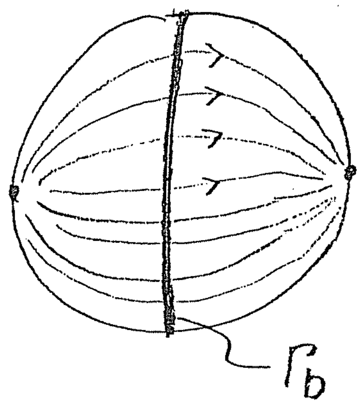
必要ならば, $\partial G$ を isotopy で動かして, 次のような $k$ が存在するようにする。

1, $k$ は Legendrian curve。

2, $k$ は $G$ の meridian 。

3, $\#(k \cap \Gamma_{\partial G}) = 2$

すると, $k$ は $G$ の中に convex な Seifert 膜 $D$ をはる。しかも, $k$ のとり方から $\Gamma_D$ は 1本の proper な arc のみからなり, 従って, $ch_\xi(D)$ を下図のように一意化できる




$\mathring{G} - D$ は $\mathring{B}^3$ と diffeo. だから, Thm. 4.1 から, $G$ に入りうる tight contact str. の一意性がいえる。

以上で Main Thm ① の証明が終了する $\square$

## §5. Symplectic Structure との関連

一般に与えられた contact 3-mfd が tight であるかどうかを 判定するのは 余り易しくない。だが, 次のような強力な定理が Eliashberg によって示されている。

<u>Thm</u> 5.1. $(M^3, \xi)$ : contact 3-mfd

$(W^4, \omega)$ : compact symplectic 4-mfd

s.t. ○ $M = \partial W$

○ $\omega|_\xi$ が nondegenerate

○ $i_X \omega \wedge \omega$ と $\alpha \wedge d\alpha$ は同じ orientation を定める。

ただし, $\begin{cases} \alpha \text{ は } \xi \text{ を定義する 1-form on } M\text{。} \\ X \text{ は 外向きの vector field} \end{cases}$

$\Longrightarrow$ このとき $\xi$ は tight 。

証明は Gromov の pseudo holomorphic curve の理論に基づく。このような $W$ は contact 3-mfd $M$ の symplectic filling と呼ばれる。例えば Ex. 1.4 において unit ball $B^4(1)$ は $(S^3, \xi)$ の symplectic filling になっている。

## §6 Open Problems

1. tight contact str を admit しないような closed ori. 3-mfd はあるか?

2. $S^3$, $S^2 \times S^1$, $T^3$ 以外の場合の tight contact str. の分類

3. Thm. 5.1 のような symplectic mfd が存在しない tight contact 3-mfd はあるか?

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— End —

# Legendrian 結び目と Bennequin の不等式について

神田 雄高 (東大数理)

本稿では tight な3次元 contact manifold に埋めこまれた Legendrian 結び目について考える。

<u>Def</u>.1　$f: S^1 \hookrightarrow (M, \xi)$ が Legendrian

$\Longleftrightarrow$　$f_*(TS^1)$ が $\xi$ に接っする。

さて、$L$ を $(M, \xi)$ に埋めこまれた Legendrian 結び目，$F$ を $L$ の Seifert 曲面とする。このとき、

<u>Def</u> 2　$tb(L, F) := L'$ と $F$ の代数的交点数。

ただし、$L'$ は $L$ を $\xi$ に横断的な非特異 vector 場に沿って少し動かしたときにできる曲線のことである。$tb(L, F)$ は，$F$ によって与えられる $L$ の framing に対して、$\xi|_L$ がどれだけねじれているかを表す数であって，正式には Thurston - Bennequin 不変量と呼ばれる。

一方、今、$\xi|_F$ に自明化を与えたとき、

<u>Def</u> 3　$r(L, F) := L$ の接 vector 場の、上の自明化に対する

回転数。

これを、rotation number と呼ぶ。

注意》 $tb(L, F) = tb(-L, F)$

$-\gamma(L, F) = \gamma(-L, F)$

注意》 Def 2 において 交点の符号は $M$ の向きづけによるが、
$M$ の向きは、$\xi$ によって自然に決まる向きにしておく。

次の Thm は Eliashberg による。

<u>Thm</u> 5 $(M, \xi)$ が tight

$\underset{\text{同値}}{\Longleftrightarrow}$ $^\forall L$ : Legendrian knot $\subset M$

$^\forall F$ : $L$ の Seifert 曲面

に対して、下の不等式が成り立つ。

$$tb(L, F) + |\gamma(L, F)| \leq -\chi(F) \quad \cdots \text{☆}$$

☆ を Bennequin の不等式と呼ぶ。歴史的に見れば、
Bennequin は $(M, \xi)$ が $(S^3, \xi_{standard})$ のとき、☆ が成立する

ことを示して，$(S^3, \xi_{standard})$ の tight 性を証明したのである。

さて，⇐) を示すのは容易である。なぜなら，$(M, \xi)$ が overtwisted disc をもつとすると，それは $tb(L, F_L) = 0$，$F_L \cong$ disc なる $(L, F_L)$ が存在することに他ならないが，＊において 左辺 $\geq 0$，右辺 $= -1$ となり矛盾するからである。

今から $(M, \xi) = (\mathbb{R}^3, dz + x dy = 0)$ として話を進める。

Def 4　結び目型 K を与えたとき，

$$\begin{cases} TB(K) := \max\limits_{[L_K]=K} tb(L_K) \\ \overline{TB}(K) := \max\limits_{[L_K]=K} tb(L_K) + |r(L_K)| \end{cases}$$

とおく。

ただし $[L]$ は $L$ の ambient isotopy class を表す。

＊より　$TB(K) \leq \overline{TB}(K) \leq 2\,\mathrm{genus}(K) - 1$ …◎　である。

右辺 $\geq -1$ であることに注意せよ。

我々の目標は，任意の正整数 $n$ に対して，

$TB(K_n) \leq -n$，$\overline{TB}(K_n) \leq -n$　となるような結び目型 $K_n$ を見付けることである。

道具は『$T^3$の tight contact 構造の分類』のところで述べた特性葉層の理論である。そこでは曲面 F が closed な場合を扱っていたが，我々はこれを F が compact かつ $\partial F$ が Legendrian という場合に拡張して用いる。

<u>Thm 6</u>　F が convex

$\underset{\text{同値}}{\Longleftrightarrow}$ $\exists \Gamma$：F における proper な curve たちの disjoint union

s.t. ① $\Gamma \pitchfork ch_\xi(F)$

② $\exists F^+, \exists F^-$ sub mfd (codim 0) $\subset F$

s.t. $F = F^+ \cup \Gamma \cup F^-$，$\Gamma = \partial \overline{F^+} = -\partial \overline{F^-}$

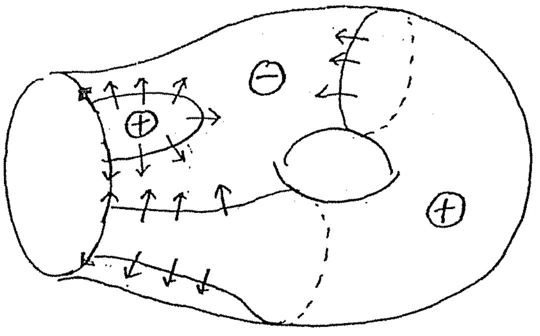
③ $\exists Y$：F 上の vector 場

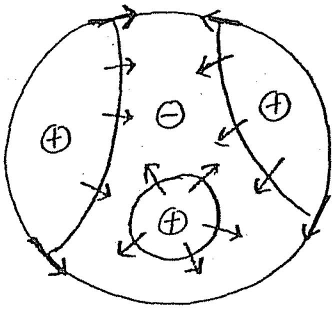
s.t. 1. Y の積分曲線のつくる foliation と $ch_\xi(F)$ が一致。

2. $\operatorname{div} Y > 0$　on $F^+$
$\phantom{2. \operatorname{div} Y} < 0$　on $F^-$

3. Y は $\partial \overline{F^+} - \partial \overline{F^+} \cap \partial F$ において $F^+$ の側から見て外向き。

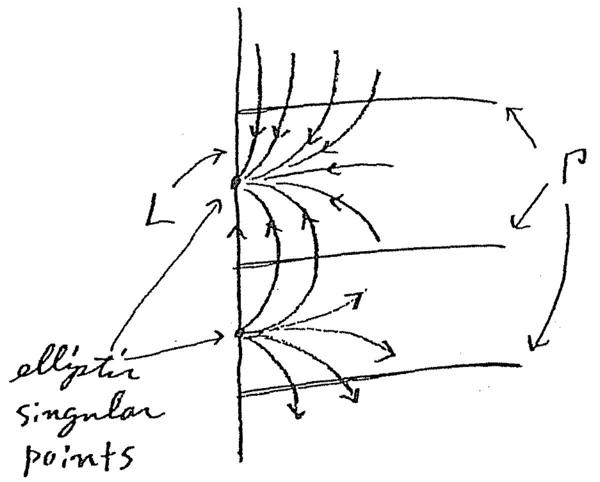
例





注)) $L := \partial F$ が Legendrian であるから, $Y$ は $L$ に接っする。

$L$ の近傍における $ch_\xi(F)$ の様子は例えば下図のようなものである。




Prop 7. $F_L$ が convex のとき

$$tb(L, F_L) = -\frac{1}{2}\#(L\cap\Gamma), \quad r(L, F_L) = \chi(F_L^+) - \chi(F_L^-)$$

注)) $F_L$ が convex に取れるためには $tb(L, F_L) \le 0$ が必要.

**Prop 8** $tb(L, F_L) \leq 0$ のとき、$F_L$ を、$\partial F_L$ を止めたまま perturb することにより、convex にできる。

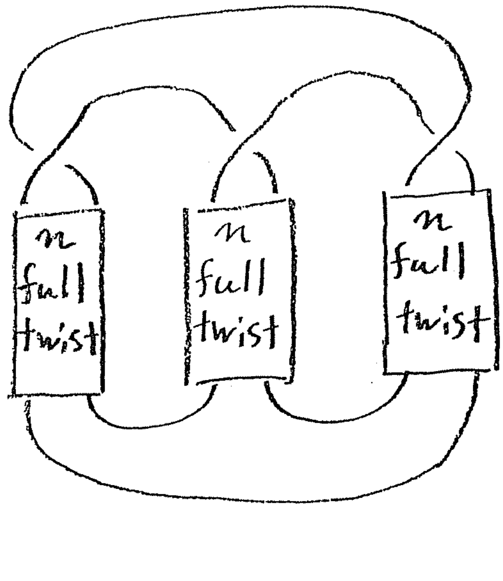
**Lemma 9** $L$ : a given Legendrian knot.
このとき $\exists L'$ : Legendrian knot s.t.
- $L$ と $L'$ は ambient isotopic
- $tb(L') + |r(L')| = tb(L) + |r(L')|$
- $tb(L') = tb(L) - 1$

以上の準備のもとに Main result について述べる。

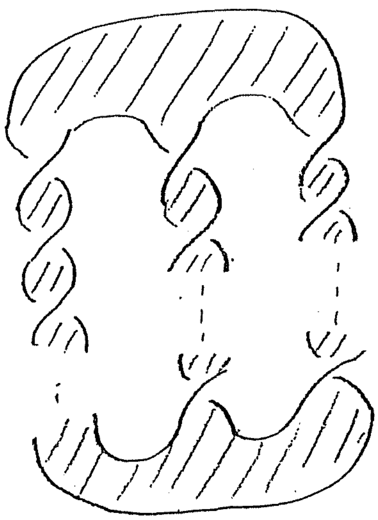
**Thm 10** $K_n$ が下図のような knot 型のとき $(n \geq 0)$

$$\overline{TB}(K_n) = -6n-5 \quad \cdots ①$$

$$TB(K_n) = -6n-6 \quad \cdots ②$$

$K_n =$ 

(ii) $K_n$ の Seifert 膜 $F_n$ として下図のようなものをとる。



$F_n \underset{\text{homeo.}}{\cong}$ one punctured torus

今，① が成り立たないとする。

このとき，Lemma 9 より，

$tb(L_n) \leq 0$，$tb(L_n) + |r(L_n)| > -6n-5$

$[L_n] = K_n$ なる $L_n$ が存在する。

$L_n$ の Seifert 膜で上図のようなものをやはり $F_n$ とかく。

このとき，Prop. 7．及び，$(\mathbb{R}^3, dz + xdy = 0)$ の tight 性などをよく考えると，次のような $F_n$ 上の simple closed curve $\ell$ が存在する。

- $\ell$ は $\partial F_n$ と $F_n$ において homotopic でない。
- $lk(\ell, \ell^*) - \frac{1}{2}\#(\ell \cap \Gamma) > 2\,\mathrm{genus}(\ell) - 1$

  ただし，$\ell^*$ は $\ell$ を $F_n$ の normal 方向に少し動かしてできる knot である。

admissible isotopy により，$\ell$ を $\ell \subset F_n$ なる Legendrian Knot として実現すると，$tb(\ell) = lk(\ell, \ell^*) - \frac{1}{2}\#(\ell \cap \Gamma)$

となるが これは Bennequin の不等式 ※ を満さず 矛盾。

よって 背理法により ①は正しい。

③についても同様。

①②の等号については、実例をつくって確かめればよい。

Reference: Ya Eliashberg "Legendrian and transversal knot in tight contact 3-manifolds" preprint.

# Hermitian Geometry on Hermitian Manifolds

– On Local Conformal Hermitian-Flatness of Hermitian Manifolds –

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## 0 Introduction

Let $M$ be a $2m$-dimensional almost Hermitian manifold with the almost complex structure $J$ and the Hermitian metric $g$. It is well-known (cf. [6], [9]) that there is a unique linear connection $D$ of $M$ such that $Dg = 0$ (*metric connection*) and $DJ = 0$ (*almost complex connection*) and that the torsion tensor $T$ is pure in the following sense:

$$T(JX, Y) = T(X, JY) \qquad \text{for any vector fields } X, Y \text{ on } M.$$

This connection is called the *Hermitian connection* (or *canonical connection*) of the almost Hermitian manifold $(M, J, g)$. In the Hermitian case, it was introduced by S.-S. Chern [4].

Balas [1] studied Hermitian manifolds of constant holomorphic sectional curvature and obtained an example of compact non-Kählerian Hermitian manifold which is Hermitian-flat. It is still unknown whether there are compact non-Kählerian Hermitian manifolds of non-zero constant holomorphic sectional curvature.

In [10], we introduced the notion of a locally conformally Hermitian-flat manifold as a complex analogue of a conformally flat Riemannian manifold. Then we derived a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian-flat in the case where the dimension is no less than 6, and we then constructed a family of examples of locally conformally Hermitian-flat metrics.

In Section 1 of this note, we recall the Hermitian connection $D$ of an almost Hermitian manifold by the *Koszul-type formula*. In Section 2, we recall the fundamental formulas for the curvature tensor $H$ of the Hermitian connection $D$ of a Hermitian manifold $(M, J, g)$.

In Section 3, we derive a necessary and sufficient condition for the product of certain normal almost contact Riemannian manifolds to be Hermitian-flat. Consequently, we obtain some examples of Hermitian-flat manifolds.

In Section 4, we recall the definition of a locally conformally Hermitian-flat manifold and derive a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian-flat in the case where the dimension is no less than 4. We then introduce a conformally invariant tensor $\mathfrak{B}$ which is naturally required from the local conformal Hermitian-flatness of a Hermitian manifold. On a locally conformally Hermitian-flat manifold, the Ricci-type tensors $Q, R, S$ and the scalar curvatures $s, \hat{s}$ have the remarkable properties. We show that, if two of the three Ricci-type tensors $Q, R$ and $S$ of a locally conformally Hermitian-flat manifold $(M, J, g)$ is either Hermitian-flat or of pointwise constant holomorphic sectional curvature.

Preliminary Remarks. 1) Throughout this paper, we always assume the differentiability of class $C^\infty$ and assume that manifolds to be connected and without boundary. We use the real dimensions for ones of manifolds. Since every 2-dimensional Hermitian manifold is Kählerian (cf. [9]), we always assume that the dimensions of (almost) Hermitian manifolds are no less than 4.

2) Given a manifold $M$, $C^\infty(M)$ denotes the space of all real valued differentiable functions on $M$ and $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on $M$.

## 1 Hermitian Connections of Almost Hermitian Manifolds

Let $M$ be an almost complex manifold with the almost complex structure $J$ and $g$ a Hermitian metric on $M$, that is, a Riemannian metric such that $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. The triple $(M, J, g)$ is called an almost Hermitian manifold. If the almost complex structure $J$ is integrable, then $(M, J, g)$ is called a Hermitian manifold. The following theorem is well-known.

**Theorem 1.1** (cf. [6], [9]) *Every almost Hermitian manifold $(M, J, g)$ admits a unique linear connection $D$ such that $Dg = 0$ and $DJ = 0$ and that the torsion tensor $T$ is pure in the following sense:*

$$T(JX, Y) = T(X, JY) \qquad \textit{for any } X, Y \in \mathfrak{X}(M).$$

The connection $D$ is called the *Hermitian connection* of $(M, J, g)$.

In our discussion of the present paper, we want the explicit expression of the Hermitian connection $D$ so that we prove Theorem 1.1 in the different way from [7] and [9]. We consider a mapping $\mathcal{V} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\mathcal{V}(X, Y) = [JX, JY] + [X, Y] - J[X, JY] + J[JX, Y] \tag{1.1}$$

for any $X, Y \in \mathfrak{X}(M)$. The proof of the following lemma is a straightforward calculation.

**Lemma 1.1** *The mapping $\mathcal{V}$ has the following properties:*

$$\mathcal{V}(X \pm Y, Z) = \mathcal{V}(X, Z) \pm \mathcal{V}(Y, Z), \quad \mathcal{V}(X, Y \pm Z) = \mathcal{V}(X, Y) \pm \mathcal{V}(X, Z), \tag{1.2}$$

$$\mathcal{V}(fX, Y) = f\mathcal{V}(X, Y), \quad \mathcal{V}(X, fY) = f\mathcal{V}(X, Y) + 2(Xf)Y + 2(JXf)JY, \tag{1.3}$$

$$\mathcal{V}(JX, JY) = \mathcal{V}(X, Y), \quad \mathcal{V}(JX, Y) = -J\mathcal{V}(X, Y), \quad \mathcal{V}(X, JY) = J\mathcal{V}(X, Y), \tag{1.4}$$

$$\mathcal{V}(X, Y) + \mathcal{V}(Y, X) = -2J[X, JY] + 2J[JX, Y], \tag{1.5}$$

$$\mathcal{V}(X, Y) - \mathcal{V}(Y, X) = 2[JX, JY] + 2[X, Y] \tag{1.6}$$

*for any $X, Y, Z \in \mathfrak{X}(M)$ and any $f \in C^\infty(M)$.*

*Proof of Theorem 1.1.* (*Existence*) For each $X, Y \in \mathfrak{X}(M)$, we define $D_X Y$ by the following equation:

$$4g(D_X Y, Z) = 2Xg(Y, Z) - 2JXg(JY, Z) + g(\mathcal{V}(X, Y), Z) - g(\mathcal{V}(X, Z), Y) \tag{1.7}$$

for any $Z \in \mathfrak{X}(M)$. By (1.2) and (1.3), it is a straightforward verification that the mapping $(X, Y) \to D_X Y$ determines a linear connection of $M$, denoted by $D$. Since the last three terms in the right hand side of (1.7) are skew-symmetric in $Y$ and $Z$, it is clear that $g(D_X Y, Z) + g(D_X Z, Y) = Xg(Y, Z)$, i.e., $Dg = 0$. To show that $DJ = 0$, it is sufficient to prove

$$D_X(JY) = JD_X Y \quad \text{for any } X, Y \in \mathfrak{X}(M).$$

This follows immediately from (1.4) and (1.7). Moreover, by (1.7) and $DJ = 0$, we have

$$D_{JX}Y = D_X(JY) - \frac{1}{2}J\mathcal{V}(X,Y). \tag{1.8}$$

From (1.8) and (1.5), we easily obtain $T(JX,Y) = T(X,JY)$.

(*Uniqueness*) It is sufficient to prove that if a linear connection $D$ satisfies $Dg = 0$, $DJ = 0$ and $T(JX,Y) = T(X,JY)$ for any $X, Y \in \mathfrak{X}(M)$, then it satisfies the equation (1.7). From $T(JX,Y) = T(X,JY)$, we have

$$g(T(X,Y),Z) + g(T(JX,JY),Z) = 0, \tag{1.9}$$

$$-g(T(X,Z),Y) - g(T(JX,JZ),Y) = 0, \tag{1.10}$$

$$g(T(X,JY),JZ) - g(T(JX,Y),JZ) = 0, \tag{1.11}$$

$$-g(T(X,JZ),JY) + g(T(JX,Z),JY) = 0 \tag{1.12}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Let us make the sum of the equations (1.9) $\sim$ (1.12). Then, from the definition of $T$ and $DJ = 0$, we obtain

$$\begin{aligned} &2g(D_XY,Z) - 2g(D_XZ,Y) \\ &\quad +2g(D_{JX}JY,Z) + 2g(D_{JX}Z,JY) - g(\mathcal{V}(X,Y),Z) + g(\mathcal{V}(X,Z),Y) = 0. \end{aligned}$$

Furthermore, by $Dg = 0$, we obtain

$$4g(D_XY,Z) - 2Xg(Y,Z) + 2JXg(JY,Z) - g(\mathcal{V}(X,Y),Z) + g(\mathcal{V}(X,Z),Y) = 0.$$

Hence $D$ satisfies (1.7). ∎

The expression of the Hermitian connection $D$ given by (1.7) is our requirement.

Let $\nabla$ be the Levi-Civita connection with respect to the same metric $g$, that is, $\nabla_XY$ is defined by the following *Koszul formula* (cf. [9]):

$$\begin{aligned} 2g(\nabla_XY,Z) = &Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &+g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. By using (1.8) and $Dg = 0$, we easily obtain

$$2g(\nabla_XY,Z) = 2g(D_XY,Z) - g(T(X,Y),Z) + g(T(Y,Z),X) - g(T(Z,X),Y). \tag{1.13}$$

Thus, if $D$ is torsion-free, then (1.13) means that $\nabla = D$. The following fact is also well-known.

**Theorem 1.2** (cf. [9]) *Let $(M,J)$ be an almost complex manifold. The almost complex structure $J$ is integrable if and only if there exists a linear connection which is almost complex and torsion-free.*

These facts yield the following well-known theorem.

**Theorem 1.3** (cf. [6], [9]) *An almost Hermitian manifold $(M,J,g)$ is a Kählerian manifold if and only if the Hermitian connection $D$ coincides with the Levi-Civita connection $\nabla$.*

## 2 Curvature Tensors

Let $(M, J, g)$ be an almost Hermitian manifold. The curvature tensor $H$ of the Hermitian connection $D$ of $M$ is defined by

$$H(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \qquad \text{for any } X,Y,Z \in \mathfrak{X}(M).$$

Furthermore, we introduce a tensor of type (0,4), denoted by the same symbol $H$, defined by

$$H(X,Y,Z,W) = g(H(Z,W)Y,X) \qquad \text{for any } X,Y,Z,W \in \mathfrak{X}(M).$$

**Lemma 2.1** (cf. [9]) *The curvature tensor $H$ of the Hermitian connection $D$ of an almost Hermitian manifold $(M, J, g)$ satisfies the following equations: For any $X,Y,Z,W \in \mathfrak{X}(M)$,*

$$H(X,Y,Z,W) = -H(Y,X,Z,W) = -H(X,Y,W,Z),$$

$$H(JX,JY,Z,W) = H(X,Y,Z,W),$$

$$\mathfrak{S}\{H(X,Y)Z\} = \mathfrak{S}\{T(T(X,Y),Z) + (D_X T)(Y,Z)\}, \quad \textit{(Bianchi's first identity)}$$

$$\mathfrak{S}\{(D_X H)(Y,Z) + H(T(X,Y),Z)\} = 0, \qquad \textit{(Bianchi's second identity)}$$

*where $\mathfrak{S}$ denotes the cyclic sum with respect to $X, Y$ and $Z$.*

By a direct computation, we have

$$\begin{aligned} &H(X,Y,JZ,JW) - H(X,Y,Z,W) \\ &\quad = -\frac{1}{2}J\mathcal{N}(Z,W)g(JX,Y) - \frac{1}{8}\mathcal{N}_1(X,Y,Z,W) + \frac{1}{8}\mathcal{N}_1(Y,X,Z,W), \end{aligned}$$

for any $X,Y,Z,W \in \mathfrak{X}(M)$, where $\mathcal{N}$ denotes the Nijenhuis tensor of $J$, i.e.,

$$\mathcal{N}(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

and

$$\begin{aligned} \mathcal{N}_1(X,Y,Z,W) &= g(\mathcal{V}(W,\mathcal{N}(Z,X)),Y) - g(\mathcal{V}(Z,\mathcal{N}(W,X)),Y) \\ &\quad +2g(\mathcal{N}(W,[Z,X]),Y) - 2g(\mathcal{N}(Z,[W,X]),Y) - 2g(\mathcal{N}(JW,[JZ,X]),Y) \\ &\quad +2g(\mathcal{N}(JZ,[JW,X]),Y) - 2g(\mathcal{N}([W,Z],X),Y) + 2g(\mathcal{N}([JW,JZ],X),Y) \\ &\quad +4g(J[J\mathcal{N}(W,Z),X],Y). \end{aligned}$$

In particular, we have

**Lemma 2.2** (cf. [6], [9]) *The curvature tensor $H$ of the Hermitian connection $D$ of a Hermitian manifold $(M, J, g)$ satisfies*

$$H(X,Y,JZ,JW) = H(X,Y,Z,W) \qquad \textit{for any } X,Y,Z,W \in \mathfrak{X}(M).$$

Let $(M,J,g)$ be a Hermitian manifold of dimension $2m$. We define three tensors $Q,R$ and $S$ which are analogous to the Ricci tensor in the Riemannian geometry. These are defined by

$$\begin{aligned}Q(X,Y) = \frac{1}{2}\sum_{\alpha=1}^{m}\{&H(e_\alpha,X,e_\alpha,Y)+H(Je_\alpha,X,Je_\alpha,Y)\\ &+H(e_\alpha,Y,e_\alpha,X)+H(Je_\alpha,Y,Je_\alpha,X)\},\end{aligned}$$

$$R(X,Y) = \sum_{\alpha=1}^{m} H(e_\alpha,Je_\alpha,X,JY),$$

$$S(X,Y) = \sum_{\alpha=1}^{m} H(X,JY,e_\alpha,Je_\alpha),$$

for any $X,Y \in \mathfrak{X}(M)$, where $\{e_1,...,e_m,Je_1,...,Je_m\}$ is a local adapted orthonormal frame field of $(M,J,g)$. Then, by Lemma 2.1 and Lemma 2.2, we have

**Lemma 2.3** *All the Ricci-type tensors $Q,R$ and $S$ defined above are symmetric and compatible with $J$, i.e.,*

$$Q(X,Y)=Q(Y,X),\quad Q(JX,JY)=Q(X,Y),$$

$$R(X,Y)=R(Y,X),\quad R(JX,JY)=R(X,Y),$$

$$S(X,Y)=S(Y,X),\quad S(JX,JY)=S(X,Y)$$

*for any $X,Y \in \mathfrak{X}(M)$.*

We can associate 2-forms $\rho_Q,\rho_R$ and $\rho_S$ with the Ricci-type tensors $Q,R$ and $S$ respectively in the usual manner:

$$\rho_Q(X,Y)=Q(X,JY),\quad \rho_R(X,Y)=R(X,JY),\quad \rho_S(X,Y)=S(X,JY)$$

for any $X,Y \in \mathfrak{X}(M)$. In particular, we then have

$$d\rho_R(X,Y,Z)=\frac{1}{3}\mathfrak{S}\{(D_X\rho_R)(Y,Z)+\rho_R(T(X,Y),Z)\}.$$

for any $X,Y,Z \in \mathfrak{X}(M)$. Furthermore, we have

$$\begin{aligned}(D_X\rho_R)(Y,Z) &= X\rho_R(Y,Z)-\rho_R(D_XY,Z)-\rho_R(Y,D_XZ)\\ &= \sum_{\alpha=1}^{m}\{Xg(H(Y,JZ)Je_\alpha,e_\alpha)-g(H(D_XY,Z)Je_\alpha,e_\alpha)\\ &\qquad -g(H(Y,D_XZ)Je_\alpha,e_\alpha)\}\\ &= \sum_{\alpha=1}^{m}\{g((D_XH)(Y,JZ)Je_\alpha,e_\alpha)+2g(H(Y,Z)D_X(Je_\alpha),e_\alpha)\}.\end{aligned}$$

However, we can show that $\sum_{\alpha=1}^{m} g(H(Y,Z)D_X(Je_\alpha),e_\alpha)=0$. In fact,

$$\begin{aligned}\sum_{\alpha=1}^{m} g(H(Y,Z)D_X(Je_\alpha),e_\alpha) = \sum_{\alpha,\beta=1}^{m}\{&g(H(Y,Z)e_\beta,e_\alpha)g(D_X(Je_\alpha),e_\beta)\\ &+g(H(Y,Z)Je_\beta,e_\alpha)g(D_X(Je_\alpha),Je_\beta)\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha,\beta=1}^{m} \{g(H(Y,Z)e_\beta, e_\alpha) g(e_\alpha, D_X(Je_\beta)) \\
&\qquad + g(H(Y,Z)e_\beta, Je_\alpha) g(Je_\alpha, D_X(Je_\beta))\} \\
&= \sum_{\beta=1}^{m} g(H(Y,Z)e_\beta, D_X(Je_\beta)) \\
&= -\sum_{\alpha=1}^{m} g(H(Y,Z)D_X(Je_\alpha), e_\alpha).
\end{aligned}
$$

Thus we have

$$
d\rho_R(X,Y,Z) = \frac{1}{3}\sum_{\alpha=1}^{m} g(\mathfrak{S}\{(D_X H)(Y,Z) + H(T(X,Y),Z)\}e_\alpha, Je_\alpha).
$$

Hence the Bianchi's second identity gives us the following

**Lemma 2.4** (cf. [9]) *The 2-form $\rho_R$ is closed.*

The 2-form $\rho_R$ is called the *Ricci form* of the Hermitian connection $D$.

Moreover, we define two *scalar* curvatures $s$ and $\hat{s}$ which are analogous to the scalar curvature in the Riemannian geometry:

$$
s = 2\sum_{\alpha=1}^{m} R(e_\alpha, e_\alpha) = 2\sum_{\alpha=1}^{m} S(e_\alpha, e_\alpha), \quad \hat{s} = 2\sum_{\alpha=1}^{m} Q(e_\alpha, e_\alpha).
$$

# 3 Hermitian-Flat Manifolds

**Definition 3.1** ([10]) We call a Hermitian manifold $(M, J, g)$ *Hermitian-flat* and $g$ a *Hermitian-flat metric* if the curvature tensor of the Hermitian connection with respect to $g$ vanishes everywhere.

Balas [1] gave an example of compact non-Kählerian Hermitian-flat manifolds.

**Example 3.1** ([1], [10]) *The Iwasawa manifold $M$* is defined by $M = G/\Gamma$, where

$$
G = \left\{ \begin{pmatrix} 1 & z^1 & z^2 \\ 0 & 1 & z^3 \\ 0 & 0 & 1 \end{pmatrix} : z^i \in \mathbf{C} \right\}, \quad \Gamma = \left\{ \begin{pmatrix} 1 & \alpha^1 & \alpha^2 \\ 0 & 1 & \alpha^3 \\ 0 & 0 & 1 \end{pmatrix} : \alpha^i \in \mathbf{Z} + \sqrt{-1}\mathbf{Z} \right\}.
$$

Since $M$ is the quotient space of a complex Lie group $G$ by a discrete subgroup $\Gamma$, it is a complex manifold. And it is also well-known that this manifold is compact. The holomorphic 1-form $\varphi = dz^2 - z^3 dz^1$ defined on $G$ is invariant under the action of $\Gamma$. Thus $\varphi$ is identified with a 1-form on $M$. Moreover we have that $d\varphi = -dz^3 \wedge dz^1 \neq 0$. Hence $M$ does not admit any Kählerian metrics. The following Hermitian metric $ds^2$ on $G$ is a Hermitian-flat metric invariant under the action of $\Gamma$. Hence $ds^2$ induces a Hermitian-flat metric on the Iwasawa manifold $M$.

$$
\begin{aligned}
ds^2 &= A_1 dz^1 d\bar{z}^1 + A_2(dz^2 - z^3 dz^1)(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) + A_3 dz^3 d\bar{z}^3 \\
&\quad + A_4 dz^1(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) + A_4(dz^2 - z^3 dz^1)d\bar{z}^1 \\
&\quad + A_5(dz^2 - z^3 dz^1)d\bar{z}^3 + A_5 dz^3(d\bar{z}^2 - \bar{z}^3 d\bar{z}^1) \\
&\quad + A_6 dz^3 d\bar{z}^1 + A_6 dz^1 d\bar{z}^3,
\end{aligned} \tag{3.1}
$$

where $A_i$ $(i = 1, ..., 6)$ are real numbers satisfying

$$A_2 > 0,\ A_1A_2 - A_4^2 > 0,\ (A_1A_2 - A_4^2)A_3 - A_1A_5^2 - A_2A_6^2 + 2A_4A_5A_6 > 0.$$

Next we shall give examples of non-compact Hermitian-flat manifolds which are the products of certain normal almost contact Riemannian manifolds. Let $M$ be an almost contact Riemannian manifold with the structure tensors $(\phi, \xi, \eta, g)$. Then we have the following relations:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi,$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Moreover, it is well-known that an almost contact structure $(\phi, \xi, \eta)$ is normal if and only if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor of $\phi$, i.e.,

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y]$$

for any $X, Y \in \mathfrak{X}(M)$.

Let $M$ and $N$ be two normal almost contact Riemannian manifolds with the structure tensors $(\phi_M, \xi_M, \eta_M, g_M)$ and $(\phi_N, \xi_N, \eta_N, g_N)$ respectively. On the product manifold $M \times N$, we define an almost complex structure $J$ as follows: Identifying $T_{(x,y)}(M \times N)$ with $T_xM \oplus T_yN$, we may express $X \in \mathfrak{X}(M \times N)$ as

$$X = X_1 + X_2,$$

where $X_1 \in \mathfrak{X}(M)$ and $X_2 \in \mathfrak{X}(N)$. We also consider a function on $M$ (resp. $N$) as a function on $M \times N$ as usual. Then an almost complex structure $J$ is defined by Morimoto [11] as

$$JX = \phi_M X_1 - \eta_N(X_2)\xi_M + \phi_N X_2 + \eta_M(X_1)\xi_N. \tag{3.2}$$

for any $X \in \mathfrak{X}(M \times N)$. This almost complex structure $J$ is integrable because both $M$ and $N$ are normal. Next we consider the product metric $g = g_M + g_N$ on the complex manifold $(M \times N, J)$. Then $g$ is compatible with $J$, that is, $(M \times N, J, g)$ is a Hermitian manifold.

Kenmotsu [8] studied a class of almost contact Riemannian manifolds which satisfy the following two conditions:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X,$$

$$\nabla_X \xi = X - \eta(X)\xi.$$

Such manifolds are called *Kenmotsu* manifolds. Kenmotsu manifolds are normal because of $[\phi, \phi] = 0$ and $d\eta = 0$. Moreover, Kenmotsu manifolds are non-compact because of $\operatorname{div} \xi = \dim M - 1$. A warped product space $\mathbf{R} \times_f \mathbf{C}^p$ is an example of Kenmotsu manifolds, where $f(t) = ce^t$ ($c$ : positive constant) is a function on $\mathbf{R}$. Here the metrics on $\mathbf{R}$ and $\mathbf{C}^p$ are the standard flat metrics. We notice that this Kenmotsu manifold $\mathbf{R} \times_f \mathbf{C}^p$ has constant sectional curvature $-1$.

Let $M$ (resp. $N$) be a Kenmotsu manifold with the structure tensors $(\phi_M, \xi_M, \eta_M, g_M)$ (resp. $(\phi_N, \xi_N, \eta_N, g_N)$). On the product manifold $M \times N$, we consider the almost complex structure $J$ defined by (3.2) and the product metric $g = g_M + g_N$. Then $(M \times N, J, g)$ is a Hermitian manifold because both $(M, \phi_M, \xi_M, \eta_M, g_M)$ and $(N, \phi_N, \xi_N, \eta_N, g_N)$ are normal. The Hermitian connection $D$ of $M \times N$ relates to the Levi-Civita connections of $M$ and $N$ as follows:

$$\begin{aligned} D_X Y = {}& {}^M\nabla_{X_1} Y_1 + {}^N\nabla_{X_2} Y_2 - \eta_M(Y_1)X_1 + \eta_N(X_2)\phi_M Y_1 + g_M(X_1, Y_1)\xi_M \\ & -\eta_N(Y_2)X_2 - \eta_M(X_1)\phi_N Y_2 + g_N(X_2, Y_2)\xi_N, \end{aligned}$$

where ${}^M\nabla$ (resp. ${}^N\nabla$) denotes the Levi-Civita connection of $M$ (resp. $N$). The curvature tensor $H$ of $D$ satisfies

$$\begin{aligned} H(X, Y, Z, W) = {}& K_M(X_1, Y_1, Z_1, W_1) + K_N(X_2, Y_2, Z_2, W_2) \\ & + g_M(X_1, Z_1)g_M(Y_1, W_1) - g_M(X_1, W_1)g_M(Y_1, Z_1) \\ & + g_N(X_2, Z_2)g_N(Y_2, W_2) - g_N(X_2, W_2)g_N(Y_2, Z_2), \end{aligned}$$

where $K_M$ (resp. $K_N$) denotes the curvature tensor of ${}^M\nabla$ (resp. ${}^N\nabla$). Evidently we obtain the following

**Theorem 3.1** *The product of two Kenmotsu manifolds $M$ and $N$ is Hermitian-flat if and only if both $M$ and $N$ have constant sectional curvature $-1$.*

# 4 Locally Conformally Hermitian-Flat Manifolds

In this section, we study Hermitian manifolds which are locally conformal to Hermitian-flat manifolds.

**Definition 4.1** ([10]) A Hermitian manifold $(M, J, g)$ is *locally conformally Hermitian-flat* if every $x \in M$ has an open neighborhood $U$ with a differentiable function $\sigma : U \to \mathbf{R}$ such that

$$g' = e^{-\sigma} g_{|U}$$

is a Hermitian-flat metric on $U$. In particular, we call $(M, J, g)$ a *globally conformally Hermitian-flat* manifold if we can take $U = M$.

**Example 4.1** Every Hermitian-flat manifold is locally conformally Hermitian-flat.

Vaisman [15] studied a class of locally conformally Kählerian manifolds whose local Kählerian metrics are flat, which are called *locally conformally Kählerian-flat* manifolds. They are automatically examples of locally conformally Hermitian-flat manifolds.

**Example 4.2** ([15]) Let $\alpha$ be any non-zero complex number with $|\alpha| \neq 1$ and fixed. The quotient space $H_\alpha^m = (\mathbf{C}^m - \{0\})/G_\alpha$, $m \geq 2$, is a complex manifold of dimension $2m$, where $G_\alpha$ is the infinite cyclic group generated by the transformation $(z^1, ..., z^m) \to (\alpha z^1, ..., \alpha z^m)$ of $\mathbf{C}^m - \{0\}$. This manifold $H_\alpha^m$ is called *a (homogeneous) Hopf manifold.* Because $H_\alpha^m$ is diffeomorphic to $S^1 \times S^{2m-1}$, $H_\alpha^m$ is compact. Here $S^1$ (resp. $S^{2m-1}$) denotes the standard 1 (resp. $(2m-1)$)-dimensional sphere. On $\mathbf{C}^m - \{0\}$, we consider a Hermitian metric

$$ds^2 = \frac{2}{\|z\|^2} \sum_{i=1}^{m} dz^i d\bar{z}^i, \tag{4.1}$$

where $\|z\|^2 = \sum_{i=1}^m z^i \bar{z}^i$. Since this metric is invariant under the action of $G_\alpha$, it induces a Hermitian metric on $H_\alpha^m$. The Hopf manifold $H_\alpha^m$ with the metric (4.1) is a locally conformally Kählerian-flat manifold.

In [10], we gave an example of the family of locally conformally Hermitian-flat metrics on the non-compact complex manifold $\mathbf{R}^{m-1} \times T^{m+1}$, where $T^{m+1}$ denotes the $(m+1)$-dimensional torus.

**Example 4.3** ([10]) Let $W$ be the $m$-times $(m \geq 2)$ direct product of $\mathbf{C}^* = \mathbf{C} - \{0\}$, i.e., $W = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$ and $\alpha$ any fixed non-zero complex number with $|\alpha| \neq 1$. Consider the transformation $g_\alpha$ of $W$ defined by $g_\alpha(z^1, ..., z^m) = (\alpha z^1, ..., \alpha z^m)$. The infinite cyclic group $G_\alpha$ generated by $g_\alpha$ acts on $W$ freely and properly discontinuously. Thus $M_\alpha = W/G_\alpha$ is a complex manifold of dimension $2m$. Moreover we can easily show that $M_\alpha$ is diffeomorphic to $\mathbf{R}^{m-1} \times T^{m+1}$. Then we can construct the family $\{ds_q^2\}$ of locally conformally Hermitian-flat metrics on $M_\alpha$ which are neither Kählerian nor locally conformally Kählerian:

$$ds_q^2 = \frac{2}{\|z\|^{2q}} \sum_i A_i \left| z^{\mu(i)} \right|^{2(q-1)} dz^i d\bar{z}^i \tag{4.2}$$

for each $(A_1, ..., A_m) \in ({}^+\mathbf{R})^m, \mu \in \mathfrak{S}_m - \{Id\}$ and $q \in \mathbf{R} - \{0, 1\}$, where $\mathfrak{S}_m$ is the permutation group of $\{1, ..., m\}$.

Let $(M, J, g)$ be a Hermitian manifold of dimension $2m$. Consider a conformal change $g' = e^{-\sigma} g$ of metric $g$ where $\sigma \in C^\infty(M)$. Denoting by $D', H'$ and $\rho_{R'}$ the Hermitian connection, the curvature tensor and the Ricci form of $g'$ respectively, we have

$$D'_X Y = D_X Y - \frac{1}{2} d\sigma(X) Y - \frac{1}{2} d^{\mathbf{C}} \sigma(X) JY, \tag{4.3}$$

$$H' = e^{-\sigma}(H - \Omega \otimes dd^{\mathbf{C}} \sigma), \tag{4.4}$$

$$\rho_{R'} = \rho_R - m dd^{\mathbf{C}} \sigma, \tag{4.5}$$

where $\Omega$ denotes the fundamental form of $(M, J, g)$, i.e., $\Omega(X, Y) = g(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$ and $d^{\mathbf{C}}$ is a differential operator (see [3]) as follows: For any $r(\geq 0)$-form $\varphi$ on $M$, we define

$$J\varphi(X_1, ..., X_r) = (-1)^r \varphi(JX_1, ..., JX_r),$$

where $X_1, ..., X_r \in \mathfrak{X}(M)$. The operator $d^{\mathbf{C}}$ is defined by

$$d^{\mathbf{C}} \varphi = -J^{-1} dJ\varphi = (-1)^r JdJ\varphi \quad \text{for any } r\text{-form } \varphi \text{ on } M.$$

From (4.4) and (4.5), we have

$$H' - \frac{1}{m} \Omega' \otimes \rho_{R'} = e^{-\sigma}(H - \frac{1}{m} \Omega \otimes \rho_R). \tag{4.6}$$

Thus we naturally obtain a tensor field $\mathfrak{B}$ defined by

$$\mathfrak{B} = H - \frac{1}{m} \Omega \otimes \rho_R. \tag{4.7}$$

Furthermore, we introduce a tensor of type (1,3), denoted by the same symbol $\mathfrak{B}$, defined by

$$g(\mathfrak{B}(X, Y)Z, W) = \mathfrak{B}(W, Z, X, Y).$$

Then we have

**Theorem 4.1** *The tensor $\mathfrak{B}$ of type $(1, 3)$ is conformally invariant.*

We are interested in locally conformally Hermitian-flat manifolds. Since the equations (4.3) $\sim$ (4.6) are valid for a local conformal change $g' = e^{-\sigma}g_{|U}$, we have that, if $H' = 0$, then $\mathfrak{B} = 0$. We claim that the converse is also true. That is, we can prove the following

**Theorem 4.2** *A Hermitian manifold $(M, J, g)$ is locally conformally Hermitian-flat if and only if the tensor $\mathfrak{B}$ vanishes everywhere on $M$.*

*Proof.* Since the Ricci form $\rho_R$ is a closed 2-form of type (1,1) (see Lemma 2.3 and Lemma 2.4), it is well-known (cf. [3]) that, on a neighborhood $U$ of every point of $M$, there exists a differentiable function $\sigma$ such that

$$\rho_R = mdd^{\mathbf{c}}\sigma.$$

By means of this function $\sigma$, we consider a local conformal change $g' = e^{-\sigma}g_{|U}$. Assume that $\mathfrak{B}$ vanishes everywhere on $M$. Then we have $\mathfrak{B}' = 0$ on $U$. Thus we obtain

$$H' = \frac{1}{m}\,\Omega' \otimes \rho_{R'} = \frac{e^{-\sigma}}{m}\,\Omega \otimes (\rho_R - mdd^{\mathbf{c}}\sigma) = 0 \quad \text{on } U.$$

Hence $(M, J, g)$ is locally conformally Hermitian-flat. The converse was claimed. ∎

**Corollary 4.1** *A locally conformally Hermitian-flat manifold is globally conformally Hermitian-flat if and only if the Ricci form is $dd^{\mathbf{c}}$-exact.*

On a Kählerian manifold or a Hermitian-flat manifold, the Ricci-type tensors $Q, R$ and $S$ coincide. We shall study the Ricci-type tensors $Q, R, S$ and the scalar curvatures $s, \hat{s}$ of a locally conformally Hermitian-flat manifold. We first prove

**Theorem 4.3** *Let $(M, J, g)$ be a locally conformally Hermitian-flat manifold. If two of the three Ricci-type tensors $Q, R$ and $S$ coincide, then $(M, J, g)$ is either Hermitian-flat or of pointwise constant holomorphic sectional curvature.*

*Proof.* Since $(M, J, g)$ is locally conformally Hermitian-flat, the curvature tensor $H$ has, by Theorem 4.2, the following form:

$$H = \frac{1}{m}\,\Omega \otimes \rho_R,$$

where $m = \dim M/2$. Then we have

$$R = mQ, \quad S = \frac{s}{2m}g, \quad s = m\hat{s}.$$

If $R = Q$, then we obtain $R = 0$, i.e., $H = 0$ because of $m \geq 2$. If $S = Q$, then we have $s = \hat{s}$. Since $m \geq 2$, we obtain $s = 0$, i.e., $0 = S = Q = R$. Hence $H = 0$. If $R = S$, then $R = \dfrac{s}{2m}g$, i.e., $\rho_R = \dfrac{s}{2m}\Omega$. Thus $H$ satisfies

$$H = \frac{s}{2m^2}\,\Omega \otimes \Omega.$$

Then the holomorphic sectional curvature $\kappa(X)$ at a unit vector $X$ satisfies

$$\kappa(X) = H(X, JX, X, JX) = \frac{s}{2m^2}\,g(X, J^2X)g(X, J^2X) = \frac{s}{2m^2},$$

that is, $(M, J, g)$ is of pointwise constant holomorphic sectional curvature. ∎

By the proof of Theorem 4.3, we have

**Corollary 4.2** *A locally conformally Hermitian-flat manifold with $Q = R = S$ is Hermitian-flat.*

**Corollary 4.3** *Every locally conformally Hermitian-flat Kählerian manifold is flat.*

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# ポアソン構造の葉層とその Godbillon-Vey 類

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## 1　動機

階数一定なポアソン構造はシンプレクティック葉層構造を与えることが知られている。一方, 横断方向向き付け可能 (transversely orientable) 葉層構造には第 2 種の特性類 Godbillon-Vey class が定まる。

“シンプレクティック葉層構造の Godbillon-Vey class はもとのポアソン構造とどのような関係にあるか, どのような影響を受けているか?” 等の素朴な疑問を感じた。

そこで, 3 次元球面上の有名な Reeb 葉層構造を与えるポアソン構造を調べた結果, その基本的なアプローチは一般次元但し余次元 1 の場合に適用可能であり, シンプレクティック葉層構造の Godbillon-Vey class はポアソン幾何の言葉で記述出来ることが判明した。その結果を, 1995 年 1 月 20 日に秋田での研究会で発表した。

その時, 水谷先生 (埼玉大学) と佐藤肇先生 (名古屋大学) に “余次元が一般でも同様の議論が可能” と教え励ましていただいた。その後, 確かに余次元が一般でも “余次元が 1 の場合の議論” を平行に進めてシンプレクティック葉層構造の Godbillon-Vey class はポアソン幾何の言葉で記述出来ることが判明したつもりである (pre-print あり)。

このテクニカルレポートは 95 年 1 月の研究会の報告である。その研究会で, 著者は OHP を利用し, 講演が早すぎ/雑であったとの反省があるので今回は 余次元 1 の場合に限って議論する。

この研究の真の動機である 3 次元球面上の Reeb 葉層構造のポアソン構造の記述は複雑なのでページ数の都合上割愛する。また, この文章の直後からは Pre-print “Foliations of Poisson structures and their Godbillon-Vey classes” からの抜粋を拙い英文のまま利用させていただく。

The main results are:

**Theorem 1.1** Let $\pi$ be a full rank Poisson structure on a manifold $M^{2m+1}$. Take a riemannian metric $<\cdot,\cdot>$ on $M$ so that $<\pi^m,\pi^m> = 1$ where $\pi^m = \overbrace{\pi\wedge\cdots\wedge\pi}^{m\text{-times}}$. Then the Godbillon-Vey class corresponds to

$$*^{-1}\left(*[\xi,\pi^m]_S \wedge *([\xi,\psi(\pi)\wedge m\pi^{m-1}]_S - [\pi^m,\psi(\xi)]_S)\right)$$

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where $[\ ,\ ]_S$ is the Schouten bracket, $*$ is the star-operator with respect to $<\cdot,\cdot>$, $\xi = *(\pi^m)$, and $\psi$ is the divergence with respect to the volume of $<\cdot,\cdot>$ for each vector field and is extended as

$$\psi(X \wedge Y) = -\psi(X)Y + \psi(Y)X - [X,Y]$$

for 2-vectors fields.

**Corollary 1.1** Under the same notation in Theorem above, if the dimension of $M$ is 3, then the Godbillon-Vey class corresponds to $-\frac{1}{2}[[\xi,\pi]_S,[\xi,\pi]_S]_S$.

## 2 Preliminary

In this section, we recall the terminology of Poisson geometry and Riemannian geometry.

**Definition 2.1 (Poisson bracket)** On the function space $C^\infty(M)$ of a manifold $M$, an operation $C^\infty(M) \times C^\infty(M) \ni (f,h) \mapsto \{f,h\} \in C^\infty(M)$ satisfying

1. $\mathbf{R}$-bi-linearity $\quad \{\lambda f + \mu g, h\} = \lambda\{f,h\} + \mu\{g,h\} \qquad (\lambda,\mu \in \mathbf{R})$
2. anti-symmetry $\quad \{g,f\} = -\{f,g\}$
3. Jacobi identity $\quad \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$
4. Leibniz formula $\quad \{f,gh\} = \{f,g\}h + g\{f,h\}$

is called a Poisson bracket on $M$. A manifold $M$ with a Poisson bracket is called a Poisson manifold.

**Definition 2.2** For each $f \in C^\infty(M)$, the Hamilton vector field of $f$ is defined by

$$H_f h = <H_f, dh> = \{f,h\}\ .$$

The distribution consisting of Hamilton vector fields $H_f$ ($f \in C^\infty(M)$) of the Poisson bracket $\{\cdot,\cdot\}$ is involutive from the fact that

$$H_{\{f,h\}} = [H_f, H_h]\ .$$

If the dimension of distribution is constant, then we have the so-called symplectic foliation of the Poisson bracket.

**Definition 2.3 (Poisson tensor)** The conditions in Definition 2.1 imply the existence of the bi-vector field $\pi$ on $M$ satisfying $<\pi, df \wedge dh> = \{f,h\}$ . $\pi$ is called a Poisson tensor of the Poisson bracket $\{\cdot,\cdot\}$.

**Definition 2.4 (cf. [1], [3])** The Schouten bracket $[\cdot,\cdot]_S$ (or the notation $[\cdot,\cdot]_{\mathrm{Schouten}}$ may be used in some case) is a homogeneous bi-derivation on $\bigwedge^*(TM)$ of degree $-1$ uniquely defined by

1. $[f,h]_{\mathrm{S}} = 0 \quad \forall f,h \in \bigwedge^0(\mathrm{T}M) = C^\infty(M)$

2. $[X,f]_{\mathrm{S}} = <X, df> = Xf \quad \forall X \in \bigwedge^1(\mathrm{T}M), f \in \bigwedge^0(\mathrm{T}M)$

3. $[X,Y]_{\mathrm{S}} = [X,Y]_{\text{Lie bracket}} \quad \forall X,Y \in \bigwedge^1(\mathrm{T}M)$

4. $[T, U \wedge W]_{\mathrm{S}} = [T,U]_{\mathrm{S}} \wedge W + (-1)^{(t-1)u} U \wedge [T,W]_{\mathrm{S}}$

5. $[T,U]_{\mathrm{S}} = (-1)^{(t-1)(u-1)+1}[U,T]_{\mathrm{S}}$

6. $(-1)^{(t-1)(w-1)}[[T,U]_{\mathrm{S}},W]_{\mathrm{S}} + (-1)^{(u-1)(t-1)}[[U,W]_{\mathrm{S}},T]_{\mathrm{S}} + (-1)^{(w-1)(u-1)}[[W,T]_{\mathrm{S}},U]_{\mathrm{S}} = 0$

where the small letter $t$ means the (usual) degree of $T$, i.e., $T \in \bigwedge^t(\mathrm{T}M)$.

**Remark 2.1** This definition of Schouten bracket is different slightly from that written in [4]. The reason may be the difference of grading of multi-vectors.

It is well-known the following facts.

**Proposition 2.1** A bi-vector field $\pi$ of $M$ defines a binary operation $\{f,h\}$ by

$$\{f,h\} := <\pi, df \wedge dh> \ = -[[\pi,f]_{\mathrm{S}},h]_{\mathrm{S}}.$$

Then $\{\cdot,\cdot\}$ satisfies Jacobi identity if and only if the Schouten bracket $[\pi,\pi]_{\mathrm{Schouten}} = 0$.

**Proposition 2.2** Let $\{X_j\}$ be a local frame field of $M$ and suppose that a bi-vector field $\pi$ is written as

$$\pi = \frac{1}{2}\sum_{i,j=1}^{n} \pi^{ij} X_i \wedge X_j, \qquad \pi^{ij} + \pi^{ji} = 0$$

locally, then

$$\begin{aligned}
[\pi,\pi]_{\mathrm{Schouten}} &= \sum_{i,j,k,\ell=1}^{n} (\pi^{ij}[X_j,\pi^{k\ell}]X_i \wedge X_k \wedge X_\ell + \pi^{ij}\pi^{k\ell}X_i \wedge [X_j,X_k] \wedge X_\ell) \\
&= \frac{2}{3!} \underset{i,j,k}{\mathfrak{S}} (\sum_s \pi^{is}[X_s,\pi^{jk}] + \sum_{s,t} \pi^{is}\lambda^j_{st}\pi^{tk}) X_i \wedge X_j \wedge X_k \ ,
\end{aligned}$$

where $[X_i,X_j] = \sum_k \lambda^k_{ij} X_k$ and the symbol $\underset{ijk}{\mathfrak{S}}$ means the cyclic sum.

In particular, if we take a local basic frame field of some local coordinate $\{x_1,\ldots,x_n\}$ of $M$, then $[\pi,\pi]_{\mathrm{Schouten}}$ is equal to

$$[\pi,\pi]^{ijk}_{\mathrm{Schouten}} = 2 \underset{i,j,k}{\mathfrak{S}} \sum_{\ell=1}^{n} \pi^{i\ell} \frac{\partial \pi^{jk}}{\partial x_\ell} \ .$$

**Remark 2.2** The rank of a Poisson structure is equal to the rank of skew-symmetric matrix $(\pi^{ij})$. And regular Poisson structures are those whose rank are constant on the whole manifold.

Let $<\cdot,\cdot>$ be a riemannian metric on $M$. Then we have the induced metric on each level of multi-vector fields or differential forms, and we use the same notation $<\cdot,\cdot>$ if necessary. Let $vol_M$ be the volume form of $<\cdot,\cdot>$. Then we get a $C^\infty(M)$-linear isomorphism $\phi : \bigwedge^u(TM) \longrightarrow \bigwedge^{n-u}(T^*M)$ by

$$<\phi(U), V> := <vol_M, U \wedge V>$$

for all $V \in \bigwedge^{n-u}(TM)$. We also have the star-operator from $\bigwedge^u(TM) \longrightarrow \bigwedge^{n-u}(TM)$ defined by the riemannian metric $<\cdot,\cdot>$ as

$$< *U, V> = <U \wedge V, base_M>$$

for all $V \in \bigwedge^{n-u}(TM)$, where $base_M \in \bigwedge^n TM$ and $<base_M, base_M> = 1$.

**Proposition 2.3** Let $T$ and $U$ be arbitrary multi-vectors. Then $\phi(T) \wedge \phi(U) = \phi(*T \wedge *U)$ holds.

## 3 Poisson tensors and foliations

Let us consider an odd dimensional manifold $M$ and $n = 2m+1 = \dim M$. Take a full rank 2-vector field $\pi$ on $M$. Then we have a 1-form $\alpha = \phi(\pi^m)$. It is well-known the next result.

**Proposition 3.1** If the 2-vector field $\pi$ is a full rank Poisson tensor on $M$, namely $[\pi,\pi]_S = 0$, then $\mathrm{Ker}\alpha = \{H_f \mid f \in C^\infty(M)\}$ and $d\alpha \wedge \alpha = 0$ hold. If dim $= 3$, then $d\alpha \wedge \alpha = 0$ implies $[\pi,\pi]_S = 0$.

**Proof:** Take some riemannian metric of $M$ so that $<\pi^m, \pi^m> = 1$ and fix it. Locally, we can find an orthonormal frame field $\{X_1, X_2, \ldots, X_n\}$ such that

$$\pi = a_1 X_1 \wedge X_2 + \frac{a_2}{2} X_3 \wedge X_4 + \cdots + \frac{a_m}{m} X_{2m-1} \wedge X_{2m}$$

Let $\{\theta^1, \theta^2, \ldots, \theta^n\}$ be the corresponding dual frame field. Then we have $\pi^m = a_1 a_2 \cdots a_m X_1 \wedge \cdots X_{2m}$ and $\alpha = \phi(\pi^m) = \theta^n$. Since we chose a riemannian metric $<\cdot,\cdot>$ so that $<\pi^m, \pi^m> = 1$, we may assume $a_1 a_2 \cdots a_m = 1$.

Let $[X_i, X_j] = \sum_{k=1}^{n} \lambda_{ij}^k X_k$ $(i,j = 1,\ldots,n)$. Then $d\theta^k = -\frac{1}{2} \sum_{i,j=1}^{n} \lambda_{ij}^k \theta^i \wedge \theta^j$.

Since $\alpha = \theta^n$, the condition $d\alpha \wedge \alpha = 0$ is equivalent to $\lambda_{ij}^n = 0$ for each $i,j = 1,\ldots,2m$.

We now study the Poisson condition $[\pi,\pi]_S$. Since

$$\begin{aligned}
[\pi,\pi]_S &= \sum_{i,j=1}^{m} [\frac{a_i}{i} X_{2i-1} \wedge X_{2i}, \frac{a_j}{j} X_{2j-1} \wedge X_{2j}]_S \\
&= \sum_{i,j=1}^{m} \Big( \frac{a_i}{i} [X_{2i-1} \wedge X_{2i}, \frac{a_j}{j}]_S \wedge X_{2j-1} \wedge X_{2j} \\
&\quad - \frac{a_j}{j} [X_{2j-1} \wedge X_{2j}, \frac{a_i}{i}]_S \wedge X_{2i-1} \wedge X_{2i} \\
&\quad + \frac{a_i}{i} \frac{a_j}{j} [X_{2i-1} \wedge X_{2i}, X_{2j-1} \wedge X_{2j}]_S \Big)
\end{aligned}$$

and the first 2 terms consist only $X_i \wedge X_j \wedge X_k$ with $i, j, k < n$, the $(i, j, n)$-component of $[\pi, \pi]_S$ is equal to that of

$$\sum_{i,j=1}^{m} \frac{a_i}{i} \frac{a_j}{j} [X_{2i-1} \wedge X_{2i}, X_{2j-1} \wedge X_{2j}]_S$$

Since

$$\begin{aligned}
[X_{2i-1} \wedge X_{2i}, X_{2j-1} \wedge X_{2j}]_S &= X_{2i-1} \wedge X_{2j-1} \wedge [X_{2i}, X_{2j}]_S - X_{2i-1} \wedge X_{2j} \wedge [X_{2i}, X_{2j-1}]_S \\
&\quad - X_{2i} \wedge X_{2j-1} \wedge [X_{2i-1}, X_{2j}]_S + X_{2i} \wedge X_{2j} \wedge [X_{2i-1}, X_{2j-1}]_S
\end{aligned}$$

we see that the term of $[\pi, \pi]_S$ involving $X_n$ is

$$\begin{aligned}
&2 \sum_{i<j\leq m} \frac{a_i}{i} \frac{a_j}{j} (X_{2i-1} \wedge X_{2j-1} \wedge \lambda^n_{2i\ 2j} X_n + X_{2i} \wedge X_{2j} \wedge \lambda^n_{2i-1\ 2j-1} X_n) \\
&-2 \sum_{i,j\leq m} \frac{a_i}{i} \frac{a_j}{j} (X_{2i-1} \wedge X_{2j} \wedge \lambda^n_{2i-1\ 2j} X_n)
\end{aligned}$$

Thus, $[\pi, \pi]_S = 0$ implies $\lambda^n_{ij} = 0$ for each $i, j = 1, 2, \ldots, 2m$ and we see that

$$d\alpha = -\sum_{i=1}^{2m} \lambda^n_{i\ n} \theta^i \wedge \theta^n$$

and get the conclusion $d\alpha \wedge \alpha = 0$.

If $n = 3$, then the term of $[\pi, \pi]_S$ involving $X_n$ is just $[\pi, \pi]_S$ and therefore $[\pi, \pi]_S = 0$ is equivalent to $d\alpha \wedge \alpha = 0$. ∎

**Remark 3.1** On $\mathbf{R}^5$, take the Cartesian coordinates $\{x_1, x_2, \ldots, x_5\}$ and let $X_j = \frac{\partial}{\partial x_j}$ for $j = 1, \ldots, 5$. Then $[X_i, X_j] = 0$ for each $(i, j = 1, \ldots, 5)$. Take the Euclidean metric $<\cdot, \cdot>$. Then 2-vector field $\pi = f(x) X_1 \wedge X_2 + \frac{1}{2f} X_3 \wedge X_4$ satisfies $\pi \wedge \pi = X_1 \wedge X_2 \wedge X_3 \wedge X_4$ and full rank. Thus, $\phi(\pi^2) = \theta^5 = dx_5$ and satisfies $d\theta^5 \wedge \theta^5 = 0$. On the other hand, by direct calculation, we have

$$\begin{aligned}
[\pi, \pi]_S &= \frac{1}{f}([X_1, f]_S X_2 \wedge X_3 \wedge X_4 - [X_2, f]_S X_1 \wedge X_3 \wedge X_4 \\
&\quad - [X_3, f]_S X_1 \wedge X_2 \wedge X_4 + [X_4, f]_S X_1 \wedge X_2 \wedge X_3)\ .
\end{aligned}$$

From this equation, we claim that there are many full-rank 2-vector field on $\mathbf{R}^5$ with $d \circ \phi(\pi^2) \wedge \phi(\pi^2) = 0$ but $[\pi, \pi]_S \neq 0$ .

**Proposition 3.2** Let $\xi := *(\pi^m)$ the 1-vector field on $M$ corresponding to $\pi$. We can define a new 1-form $\beta = \phi([\xi, \pi^m])$ on $M$. Then $\beta$ satisfies $d\alpha = \alpha \wedge \beta$ for the 1-form $\alpha$ in Proposition 3.1.

**Proof:** We use the same notation in Proposition 3.1. Since $\pi^m = X_1 \wedge \cdots \wedge X_{2m}$, we see that $\xi = *(\pi^m) = X_n$. Direct calculation shows that

$$\begin{aligned}
[\xi, \pi^m]_S &= [X_n, X_1 \wedge X_2 \wedge \cdots \wedge X_{2m}]_S \\
&= [X_n, X_1] \wedge X_2 \wedge \cdots \wedge X_{2m} + X_1 \wedge [X_n, X_2] \wedge \cdots \wedge X_{2m} \\
&\quad + \cdots + X_1 \wedge X_2 \wedge \cdots \wedge [X_n, X_{2m}] \\
&= \sum_{j_1}^{n} \lambda_{n\ 1}^{j_1} X_{j_1} \wedge X_2 \wedge \cdots \wedge X_{2m} + \sum_{j_2}^{n} X_1 \wedge \lambda_{n\ 2}^{j_2} \wedge \cdots \wedge X_{2m} \\
&\quad + \cdots + \sum_{j_{2m}}^{n} X_1 \wedge X_2 \wedge \cdots \wedge \lambda_{n\ 2m}^{j_{2m}} X_{j_{2m}} \\
&= (\sum_{j}^{2m} \lambda_{n\ j}^{j}) X_1 \wedge X_2 \wedge \cdots \wedge X_{2m} \\
&\quad + \lambda_{n\ 1}^{n} X_n \wedge X_2 \wedge \cdots \wedge X_{2m} + \lambda_{n\ 2}^{n} X_1 \wedge X_n \wedge \cdots \wedge X_{2m} \\
&\quad + \cdots + \lambda_{n\ 2m}^{n} X_1 \wedge X_2 \wedge \cdots \wedge X_{2m-1} \wedge X_n
\end{aligned}$$

Thus we have

$$\phi[\xi, \pi^m]_S = (\sum_{j}^{2m} \lambda_{n\ j}^{j}) \theta^n - \sum_{j}^{2m} \lambda_{n\ j}^{n} \theta^j .$$

It holds thereby

$$d\alpha = \alpha \wedge \phi[\xi, \pi^m]_S .$$

∎

There is a secondary characteristic class called Godbillon-Vey class, expressed by 3-form corresponding to each codimension 1 foliation. We recall the definition of Godbillon-Vey class in accordance to [2]. Let $\{\tau_\mu\}$ be a family of local 1-forms of some codimension 1 distribution of $M$. The integrability condition for the given codimension 1 distribution is equivalent to $d\tau_\mu \wedge \tau_\mu = 0$ and this is equivalent to $d\tau_\mu = \tau_\mu \wedge \Theta$ for some global 1-form $\Theta$. There is some ambiguity in choosing $\Theta$ but closed 3-form $\Theta \wedge d\Theta$ is unique up to exact 3-form. Thus, the cohomology class $[\Theta \wedge d\Theta]$ is uniquely determined and is called Godbillon-Vey class of the given foliation.
From Proposition 3.2, it turns out that $\phi([\xi, \pi^m]_S) \wedge d(\phi[\xi, \pi^m]_S)$ express the Godbillon-Vey class. We would like to find this class in the context of Poisson geometry. By this reason, we push back $\phi([\xi, \pi^m]_S) \wedge d(\phi[\xi, \pi^m]_S)$ by $\phi$.
**Lemma 3.1** $\psi := \phi^{-1} \circ d \circ \phi$ is expressed by the Schouten bracket recursively as

$$\psi(T \wedge X) = [X, T]_S + \psi(X)T - \psi(T) \wedge X$$

where $X$ is a 1-vector field, $T$ is an arbitrary multi-vector field, and $\psi(X)$ is the divergence of $X$ with respect to the volume form $vol_M$ of the riemannian metric $g$. If $Q$ is a 2-vector field, then

$$\psi(T \wedge Q) = [T, Q]_S + \psi(T) \wedge Q + T \wedge \psi(Q)$$

holds for an arbitrary multi-vector field $T$.

**Proof:** Let us remember the definition of $\phi$. $<\phi(T), W> := <vol_M, T\wedge W> = <\iota_T vol_M, W>$. Thus,

$$\begin{aligned}
\psi(T\wedge X) &= \phi^{-1}\circ d\circ\phi(T\wedge X) = \phi^{-1}\circ d\circ\iota_{T\wedge X}vol_M \\
&= \phi^{-1}\circ d\circ\iota_X\circ\iota_T vol_M = \phi^{-1}\circ(\mathcal{L}_X - \iota_X\circ d)\circ\iota_T vol_M \\
&= \phi^{-1}\circ\mathcal{L}_X\circ\iota_T vol_M - \phi^{-1}\circ\iota_X\circ d\circ\iota_T vol_M \\
&= \phi^{-1}(\iota_{[X,T]_{\mathrm{S}}}vol_M + div(X)\iota_T vol_M) - \phi^{-1}\circ\iota_X\circ\phi\circ\phi^{-1}\circ d\circ\phi(T) \\
&= [X,T]_{\mathrm{S}} + div(X)T - \psi(T)\wedge X
\end{aligned}$$

∎

**Remark 3.2** If we change our view point of the equations in this Lemma 3.1, the Schouten bracket is characterized by some volume form and the mapping $\psi$. This idea is shown in [1].

It turned out the Godbillon-Vey class of foliations defined by our full rank Poisson structure $\pi$ is expressed by Godbillon-Vey form $\phi([\xi,\pi^m]_{\mathrm{S}})\wedge d(\phi[\xi,\pi^m]_{\mathrm{S}})$ on $M$.
Using the formula in Proposition 2.3, the pull back of the Godbillon-Vey form is

$$*^{-1}(*[\xi,\pi^m]_{\mathrm{S}}\wedge *\psi[\xi,\pi^m]_{\mathrm{S}})$$

We simplify $\psi[\xi,\pi^m]_{\mathrm{S}}$ using the fact that $\pi$ is a Poisson tensor. Since the Schouten bracket is derivation, we have

$$[\xi,\pi^m]_{\mathrm{S}} = [\xi,\pi]_{\mathrm{S}}\wedge m\pi^{m-1} = m\pi^{m-1}\wedge[\xi,\pi]_{\mathrm{S}}$$

Using Lemma 3.1, we see that

$$\begin{aligned}
\psi[\xi,\pi^m]_{\mathrm{S}} &= \psi(m\pi^{m-1}\wedge[\xi,\pi]_{\mathrm{S}}) \\
&= [m\pi^{m-1},[\xi,\pi]_{\mathrm{S}}]_{\mathrm{S}} + \psi(m\pi^{m-1})\wedge[\xi,\pi]_{\mathrm{S}} + m\pi^{m-1}\wedge\psi([\xi,\pi]_{\mathrm{S}})
\end{aligned}$$

Since $[\pi,\pi]_{\mathrm{S}}=0$, we have $[\pi,[\xi,\pi]_{\mathrm{S}}]_{\mathrm{S}}=0$ by the generalized Jacobi identity of Schouten bracket, and also we have $\psi(\pi^k) = k\pi^{k-1}\wedge\psi(\pi)$ for each $k$. Continuing our simplification, we see

$$\begin{aligned}
\psi[\xi,\pi^m]_{\mathrm{S}} &= m(m-1)\pi^{m-2}\wedge\psi(\pi)\wedge[\xi,\pi]_{\mathrm{S}} + m\pi^{m-1}\wedge\psi([\xi,\pi]_{\mathrm{S}}) \\
&= \psi(\pi)\wedge[\xi,m\pi^{m-1}]_{\mathrm{S}} + m\pi^{m-1}\wedge\psi([\xi,\pi]_{\mathrm{S}}) \\
&= \psi(\pi)\wedge[\xi,m\pi^{m-1}]_{\mathrm{S}} + m\pi^{m-1}\wedge(-[\pi,\psi(\xi)]_{\mathrm{S}} + [\psi(\pi),\xi]_{\mathrm{S}}) \\
&= [\xi,\psi(\pi)\wedge m\pi^{m-1}]_{\mathrm{S}} - [\pi^m,\psi(\xi)]_{\mathrm{S}}
\end{aligned}$$

Thus, we have the following result.

**Theorem 3.3** Let $\pi$ be a full rank Poisson structure on a manifold $M^{2m+1}$. Take a riemannian metric $<\cdot,\cdot>$ on $M$ so that $<\pi^m,\pi^m> = 1$. Then the Godbillon-Vey class corresponds to

$$*^{-1}\left(*[\xi,\pi^m]_S \wedge *([\xi,\psi(\pi)\wedge m\pi^{m-1}]_S - [\pi^m,\psi(\xi)]_S)\right)$$

where $[\ ,\ ]_S$ is the Schouten bracket, $*$ is the star-operator with respect to $<\cdot,\cdot>$, $\xi = *(\pi^m)$, and $\psi$ is the divergence with respect to the volume of $<\cdot,\cdot>$ for each vector field and is extended for 2-vector fields as

$$\psi(X\wedge Y) = -\psi(X)Y + \psi(Y)X - [X,Y]$$

where $X$, $Y$ are 1-vectors fields.

## 4 3-dimensional case

If dim$=3$, then the situation is rather simple. For instance, it holds that

$$*U\wedge *V = U\wedge V \qquad \text{for} \quad U\in \bigwedge^1(TM), \quad V\in\bigwedge^2(TM)$$

Thus, we shall consider the 3-vector field $[\xi,\pi]_S\wedge\psi[\xi,\pi]_S$. Since $[\xi,\pi]_S$ is a 2-vector field and the manifold $M$ is 3-dimensional, we see that $[\xi,\pi]_S\wedge[\xi,\pi]_S = 0$. Applying the map $\psi$ to this trivial equation, it turns out

$$\begin{aligned} 0 &= \psi([\xi,\pi]_S\wedge[\xi,\pi]_S) \\ &= [[\xi,\pi]_S,[\xi,\pi]_S]_S + 2\psi([\xi,\pi]_S)\wedge[\xi,\pi]_S \end{aligned}$$

We therefore have the special result for 3-dimensional manifolds.

**Theorem 4.1** Consider a nowhere vanishing Poisson structure $\pi$ on 3-dimensional manifold $M$. Take a riemannian metric $<\cdot,\cdot>$ on $M$ so that $<\pi,\pi> = 1$. Let $\xi = *\pi$, where $*$ is the star-operator with respect to $<\cdot,\cdot>$. Then the Godbillon-Vey form corresponds to $-\frac{1}{2}[[\xi,\pi]_S,[\xi,\pi]_S]_S$.

## 5 Examples

As concrete examples of 3-manifolds, we will consider 3-sphere $S^3$, $SL(2,\mathbf{R})$, and study their foliations.

### 5.1 Reeb foliation of 3-sphere

Since $S^3$ is interpreted as a Lie group $SU(2)$, we can find some global frame field $\{X_1,X_2,X_3\}$, where the coefficient $\lambda^i_{jk}$ are constant on $S^3$. In particular, we can take the global frame field $\{X_1,X_2,X_3\}$ corresponding with quaternionic $i$-, $j$-, $k$-multiplication, whose Lie bracket relations are

$$[X_1,X_2] = 2X_3, \quad [X_2,X_3] = 2X_1, \quad [X_3,X_1] = 2X_2\ .$$

Take a 2-vector field $\pi = p_1 X_2 \wedge X_3 + p_2 X_3 \wedge X_1 + p_3 X_1 \wedge X_2$. It is easy to write down the Poisson condition for $\pi$ and it is given by

$$\begin{aligned} p_1([X_2, p_3] - [X_3, p_2]) &+ p_2([X_3, p_1] - [X_1, p_3]) \\ &+ p_3([X_1, p_2] - [X_2, p_1]) - 2\left({p_1}^2 + {p_2}^2 + {p_3}^2\right) = 0 \ . \end{aligned} \qquad (1)$$

But it seems not so easy to find non-zero solution for the equation (1). Fortunately, there are many codimension 1 smooth foliations on $S^3$ found or created by differential topologists (foliators). Thus, we say that there are many non-trivial solutions for the Poisson condition (1) described above. In deed, we shall find suitable Poisson tensors associated with some famous foliations like as Reeb foliation.

Since $S^3$ has the Hopf fibration over $S^2$, it is known that $S^3$ can be decomposed as two copies of $D^2 \times S^1$. 以下, Reeb 葉層構造のポアソンテンソルを具体的に書き下す作業については省略するが, 実際行ったことは以下の項目である。

ポアソンテンソルを具体的に書き下した。

そのポアソンテンソルを用いてスカウテン括弧積を計算。

その結果 Godbillon-Vey class がゼロとなり既知の結果と整合することがわかった。

### 5.2 Anosov foliations of $SL(2, \mathbf{R})$

Let $X_1$, $X_2$, $X_3$ be the left-invariant vector field of $SL(2, \mathbf{R})$ corresponding to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. Then we have the relation

$$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1 \ .$$

Let us consider a left-invariant 2-vector field $\pi = a_1 X_2 \wedge X_3 + a_2 X_3 \wedge X_1 + a_3 X_1 \wedge X_2$, where $a_1, a_2, a_3$ are constant. Then

$$[\pi, \pi]_S = 2({a_1}^2 + 4a_2 a_3) X_1 \wedge X_2 \wedge X_3$$

holds and $\pi$ is a Poisson tensor if and only if ${a_1}^2 + 4a_2a_3 = 0$. Take a left-invariant riemannian metric $<\cdot, \cdot>$ defined by $<X_j, X_k> = \delta_{jk}$. Then $<\pi, \pi> = 1$ if and only if ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$. Then $\xi = *\pi = a_1 X_1 + a_2 X_2 + a_3 X_3$. If ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$ and ${a_1}^2 + 4a_2a_3 = 0$, then $\pi$ above is a Poisson tensor of rank 2 and the corresponding foliation is an Anosov foliation of $SL(2, \mathbf{R})$. Since

$$[\xi, \pi]_S = (-a_1a_2 + 2a_3a_1) X_1 \wedge X_2 + 2({a_2}^2 - {a_3}^2) X_2 \wedge X_3 + (a_3a_1 - 2a_1a_2) X_3 \wedge X_1$$

and

$$[[\xi, \pi]_S, [\xi, \pi]_S]_S = 8({a_2}^2 - 4a_2a_3 + {a_3}^2)^2 X_1 \wedge X_2 \wedge X_3$$

under the condition ${a_1}^2 + 4a_2a_3 = 0$, we see that the Godbillon-Vey class with respect to $\pi$ is $-4 \times$ canonical volume form.

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# 可逆非斉次な振動型積分変換から構成される無限次元 Lie 群と $S^{2n-1}$ 上の接触変換

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## 1 Introduction.

量子力学における基礎方程式として Schrödinger 方程式

$$\frac{d}{dt}\Phi_t = \frac{\sqrt{-1}}{\hbar}\hat{H}_t\Phi_t, \quad \Phi_0 = \mathrm{id}. \tag{1.1}$$

があるが、Feynman は次の式

$$\int_{\Omega(t,x,y)} \exp\frac{i}{\hbar}S(t,\gamma)\,\mathcal{D}[\gamma], \tag{1.2}$$

によって (1.1) の基本解が構成されると主張した (cf. [FH])。（ここで、$\hat{H}_t$ は time-dependent Hamiltonian operator であり、$S$ はその母関数である。また、$\Omega(t,x,y)$ は $x$ から出発して $t$ 秒後に $y$ に到達する paths 全体からなる集合で、$\mathcal{D}[\gamma]$ は $\Omega(t,x,y)$ の上の "measure" である）。

現在では (1.2) は Feynman path integral と呼ばれているがこれを数学的に厳密に取り扱おうとすると、たちまち種々の困難に遭遇する。例えば、$\Omega(t,x,y)$ 上の measure $\mathcal{D}[\gamma]$ が構成できるか？等。多くの数学者 ( cf.[F2], [F3], [Ki], [KK], [AH], [IM], [N2], [I1], [I2]) によって様々な解決策が考案されてきたが、本稿では無限次元 Lie 群上の product integral(cf. [N1], [Om], [OMYK]) という観点から (1.1), (1.2) について考えてみたい。雰囲気だけ述べれば、(1.1) を無限次元 Lie 群上の (time-dependent) ベクトル場に関する積分曲線の方程式、また (1.2) を product integral と見なしたいのである。そこで、次節において、最も単純な場合について考察することで我々の目標を鮮明にしたい。

## 2 目標

先ず、symplectic 群

$$Sp(n,\mathbb{R}) = \left\{ g \in GL(2n,\mathbb{R}) : {}^t g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \tag{2.1}$$

と、metaplectic 群 $Mp(n)$ について考える。

ここで $Mp(n)$ は次のような operator のなす群である。

(2.2)

$$Mp(n) = \left\{ I(a_{\pm}(\phi),\phi) = \text{os-}\int a_{\pm}(\phi) e^{\frac{\sqrt{-1}}{\hbar}\phi(x,\xi,y)} \sqcup(y)\,\bar{d}y\,\bar{d}\xi \;\middle|\; \phi = \frac{1}{2}(x,\xi,y)\begin{pmatrix} A & B & C \\ {}^tB & D & E \\ {}^tC & {}^tE & F \end{pmatrix}\begin{pmatrix} x \\ \xi \\ y \end{pmatrix}, \quad \begin{pmatrix} B & C \\ D & E \end{pmatrix} : \text{regular} \quad A, D, F : \text{symmetric} \right\}$$

ここで os-$\int$ は oscillatory integral transformation(cf.[Hö], [AF]) を意味する。また $đx = (2\pi\hbar)^{-\frac{n}{2}}dx$、そして $a_+(\phi)$, $a_-(\phi)$ は $\det\begin{pmatrix} B & C \\ D & E \end{pmatrix}$ によって定まる定数である。

この時

$$\begin{array}{rcccl} & \pi & : Mp(n) & \to & Sp(n,\mathbb{R}) \\ (2.3) & & \Cup & & \Cup \\ & & I(a_\pm(\phi),\phi) & \mapsto & \varphi_\phi \end{array}$$

とすれば

$$(\#) \quad 0 \to \{\pm 1\} \to Mp(n) \xrightarrow{\pi} Sp(n,\mathbb{R}) \to 1 \text{ ; exact}$$

がなりたつ。

ここで $\varphi_\phi$ は $\phi$ により定まる方程式

$$(2.4) \qquad \partial_x\phi(x,\xi,y) - \zeta = 0, \quad \partial_\xi(x,\xi,y) = 0, \quad \partial_y\phi(x,\xi,y) + \eta = 0$$

を $(y,\eta)$ について $(x,\zeta)$ で解いて得られる canonical diffeomorphism であり、象徴的に

$$(2.5) \qquad \begin{pmatrix} x \\ \partial_x\phi \end{pmatrix} \mapsto \begin{pmatrix} y \\ -\partial_y\phi \end{pmatrix} \text{ under } \partial_\xi\phi = 0$$

と書かれる。

そして、たいせつな事は $Mp(n)$ 次の表の様な意味において $Sp(n,\mathbb{R})$ の量子版と思うことが出来ると言うことである。

| | Hamiltonian | 方程式 | 解 及び 群 |
|---|---|---|---|
| 古典論 | $H_t(x,\xi)$<br>$(x,\xi)$ に関しては 2 次多項式 | canonical 方程式<br>$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_{H_t}\begin{pmatrix} q \\ p \end{pmatrix}$ | product integral<br>$\prod_0^t e^{X_\tau d\tau} \in Sp(n,\mathbb{R})$ |
| 量子論 | $\hat{H}_t(x, \frac{\sqrt{-1}}{\hbar}\partial_x)$<br>2 次多項式が symbol | Schrödinger 方程式<br>$\frac{d}{dt}\Phi_t = \frac{\sqrt{-1}}{\hbar}\hat{H}_t \cdot \Phi_t$ | path integral<br>$\prod_0^t e^{\frac{\sqrt{-1}}{\hbar}\hat{H}_\tau d\tau} \in Mp(n)$ |

このことを念頭に置き次のようなことを考えよう。

古典レベルの群を $Sp(n,\mathbb{R})$ から球面 $S^{2n-1}$ における接触変換群の単位連結成分 $\mathcal{D}_\omega(S^{2n-1})_0$ に置き換えた時、$Mp(n)$ に相当する量子レベルの群、つまり $\mathcal{D}_\omega(S^{2n-1})_0$ の量子版（$GF^0$ と記しておく）を見つけよう。更に $GF^0$ で先ほどの表が書けるようにしよう。

このためには何かうまい無限次元 Lie 群 $GF^0$ を見つけて

- Schrödinger equation を time-dependent right-invariant vector field $\frac{\sqrt{-1}}{\hbar}\hat{H}_t \in Lie(GF^0)$ に関する integral curve の方程式として捉え、その解が product integral $\prod_0^t e^{\frac{\sqrt{-1}}{\hbar}\hat{H}_\tau d\tau}$ によって構成される。（regularity といい、条件（R）と記す。）

ようにして, 更に $(\#)$ に当たる short exact sequence を作ればよい。

## 3　無限次元 Lie 群の一般論と注意.

有限次元 Lie 群 が $\mathbb{R}^m$ を model space としているように、Banach space を model space とする $C^\infty$-Lie 群 を *Banach-Lie* 群、Fréchet space を model space とする $C^\infty$-Lie 群 を *Fréchet-Lie* 群 と言う。

Banach-Lie 群においては様々な基本的定理が有限次元の時と同様に成立する。ところが、一般の Fréchet space 上では (微分可能性は有限次元のときと同様に定義されはするものの)、陰関数定理や常微分方程式の解の存在定理、product integral の収束といった基本的定理は一般には成立しない。(基本列になっているかどうかを判定するのが難しいので、折角備わっている完備性が役にたっていると言うことがすぐには分からない。) 以上が一般的な枠組である。したがってこれから構成すべき Fréchet-Lie 群 $GF^0$ についても先の条件 (R) が当然のごとく得られている訳ではなく調べてみなくてはならないことなのである。(条件 (R) をみたす Fréchet-Lie 群は regular Fréchet-Lie 群 と呼ばれている。[OMYK])

ここで、いくつか既に知られている例を挙げておこう。Banach-Lie 群 の例として invertible な bounded operator 全体として構成される Banach-Lie 群 $B(L^2(\mathbb{R}^n))$ がある。一方、Fréchet-Lie 群 の例としては compact manifold $M$ 上の diffeomorphism 全体のなす群や compact symplectic manifold $M$ 上の symplectic diffeomorphism 全体のなす群 $\mathcal{D}_\Omega(M)$、更に compact contact manifold $M$ 上の contact diffeomorphism 全体のなす 群 $\mathcal{D}_\omega(M)$ 等がある。実はこれらの diffeomorphism のなす群は regular Fréchet-Lie 群であることも知られている ([Om] を参照)。従って先ほど挙げた $\mathcal{D}_\omega(S^{2n-1})$ も regular Fréchet-Lie 群 なのである。

ところで、Schrödinger equation を Neumann 流の定式化で捉えれば

$$\begin{array}{ccc} \text{Lie algebra} & \longleftrightarrow & \{\text{unbounded skew adjoint}\} \\ \text{Lie group} & \longleftrightarrow & U(L^2) \end{array}$$

となるのであろうが、この時 {unbounded skew adjoint} の topology についてどうしたらよいのであろうか？そして、Stone の定理は

$$\frac{d}{dt}e^{\frac{\sqrt{-1}}{\hbar}\hat{H}}u = \frac{\sqrt{-1}}{\hbar}\hat{H}e^{\frac{\sqrt{-1}}{\hbar}\hat{H}}u \qquad (\forall u \in L^2)$$

を主張しているのであり、

$$\frac{d}{dt}e^{\frac{\sqrt{-1}}{\hbar}\hat{H}} = \frac{\sqrt{-1}}{\hbar}\hat{H}e^{\frac{\sqrt{-1}}{\hbar}\hat{H}}$$

を主張しているのではない (位相が異なる) と言う問題があることにも注意を要す。

## 4　母関数について.

この節では、接触変換と正準変換、そして母関数について考えてみよう。

まず、以下では接触変換 $\tilde{\varphi}$ は $\mathcal{D}_\omega(S^{2n-1})$ の 単位元に十分近いとしておく。ここで、$\mathcal{D}_\omega(S^{2n-1})$ は $S^{2n-1}$ 上 の $C^\infty$ 接触変換のなす 無限次元 (SILB-)Lie 群 (cf. [Om]) であり、$\mathcal{D}_\omega(S^{2n-1})_0$ は $\mathcal{D}_\omega(S^{2n-1})$ の単位連結成分を意味している。$\hat{U}$ は 単位元の十分小さい star 近傍。

まず、任意の $(z_1,\cdots,z_k)\in\mathbb{K}^k$ にたいして,

$$\|z_1\cdots,z_k\| = \Big\{\sum_{i=1}^{k}|z_i|^2\Big\}^{\frac{1}{2}}.$$

とおく。特に、

$$\rho=\sqrt{x_1^2+\cdots+x_n^2+\xi_1^2+\cdots+\xi_n^2},\qquad \omega\in S^{2n-1},\qquad \langle x;\xi\rangle=\sqrt{1+\rho^2}.$$

となる。

次に、 cut off function $\kappa\in C_{\mathbb{R}}^{\infty}(\mathbb{R}^{2n})$ を $0\le\kappa(x,\xi)\le 1$ であって

$$\kappa(x,\xi)=\begin{cases}0 & (\text{if } \rho\le\frac{1}{2})\\ 1 & (\text{if } \rho\ge 1)\end{cases}. \tag{4.1}$$

をみたすものとする。

$\tilde{\varphi}_t$ を $\hat{U}$ 内の曲線で $\tilde{\varphi}_0=$id, $\tilde{\varphi}_1=\tilde{\varphi}$ なるものとしよう。$\tilde{\varphi}_t\in\mathcal{D}_\omega(S^{2n-1})_0$ なので、 $\tilde{\varphi}_t^*\theta=h_t\theta$ なる $h_t\in C^\infty(S^{2n-1})$ がある。ただしここで、 $\theta$ は $S^{2n-1}$ 上の standard contact 1-form である。$\tilde{\varphi}_t^*\theta$ は $\theta$ の $\tilde{\varphi}_t$ による pull back である。

以上の記号をもちいて、接触変換 $\tilde{\varphi}$ に対応する正準変換 $\varphi$ を以下のように定義する。

$$\begin{array}{ccc}\mathbb{R}^{2n}=\mathbb{R}_+\times S^{2n-1} & \xrightarrow{\varphi} & \mathbb{R}_+\times S^{2n-1}\\ \Cup & & \Cup\\ (\rho,\omega) & \longmapsto & (f(\rho,\omega),\tilde{\varphi}_{\kappa(2\rho)}(\omega)),\end{array} \tag{4.2}$$

ここで、

$$f(\rho,\omega)=\frac{\rho}{h_{\kappa(2\rho)}(\omega)}. \tag{4.3}$$

次に Poincaré's lemma をもちいることによって、 $\varphi$ に対応する母関数を次のように構成する。

$$\begin{aligned}S_\varphi&=\int_0^1\Big\{\bar{\xi}(t\bar{x},t\xi)\bar{x}+x(t\bar{x},t\xi)\xi\Big\}dt,\\ \phi_\varphi&=S_\varphi-\xi\cdot x\end{aligned} \tag{4.4}$$

ここでは Lagrangian subspace $\mathbb{R}^{2n}_{(\bar{x},\xi)}$ を 正準変換のグラフ (submanifold) $\chi_\phi$ の chart として使っている。つまり、関係式

$$\begin{pmatrix}\bar{x}\\ \bar{\xi}\end{pmatrix}=\varphi\begin{pmatrix}x\\ \xi\end{pmatrix} \tag{4.5}$$

から $x=x(\bar{x},\xi),\bar{\xi}=\bar{\xi}(\bar{x},\xi)$ と解き $(\bar{x},\xi)$ を manifold $\chi_\varphi$ 上の coordinate として用いているのである。

(4.6)

| | |
|---|---|
| contact diffeomorphism | $\tilde{\varphi}_\phi$ |
| canonical diffeomorphism | $\varphi_\phi$ |
| generating function | $\phi_{\tilde{\varphi}},\ \phi_\varphi$ |

以下では、接触変換、 contact diffeomorphism、 contact transformation また正準変換、 canonical diffeomorphism、 canonical transformation と言う用語を混同して用いる。

## 5 ΨDO と OIT.

ここでは、ΨDO(pseudo-differential operator) と OIT(oscillatory integral transformation) について簡単に復習しておくことにする。

まず ΨDO とは以下のような形をした積分変換である。

$$(5.1)\qquad (P(a)u)(\bar{x}) = \iint a(\bar{x},\xi)e^{(\bar{x}-x)\cdot\xi}u(x)dxd\xi$$

ここで、$a(\bar{x},\xi)$ は ΨDO の symbol と呼ばれる複素数値関数である。通常は symbol class と呼ばれる関数空間の元である。(cf.[Ku])

次に $e$ の肩の $(\bar{x}-x)\cdot\xi$ を $\phi(\bar{x},\xi,x)$ (phase function、generating function と呼ばれる) に変えて得られる積分変換を oscillatory integral transformation という。(積分が意味を持つためには $\phi$ はいくつか条件をみたさなければならない
(cf.[AF]))。以下本稿では $\phi = S_\varphi + f - \xi\cdot x$ として考えることが多い。そして、そのときには $I(a,\phi)$ を $F(a,f,\varphi)$ とも書く。

次に、ΨDO どうしの積（合成）については以下が成立する（非 Weyl 型 *- 積、Moyal- 積）。

$$(5.2)\qquad \begin{aligned} &P(a)\circ P(b) = P(a*b)\\ &a*b \sim \sum \frac{1}{\alpha!}\left(\frac{\hbar}{\sqrt{-1}}\right)\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}a\cdot\frac{\partial^{|\alpha|}}{\partial x^\alpha}b \end{aligned}$$

これは、次のようにして導かれる。

$$\begin{aligned} &\iint\iint a(\bar{\bar{x}},\bar{\xi})b(\bar{x},\xi)e^{(\bar{\bar{x}}-\bar{x})\cdot\bar{\xi}+(\bar{x}-x)\cdot\xi}u(x)dxd\xi d\bar{x}d\bar{\xi}\\ =&\iint\Big\{\iint a(\bar{\bar{x}},\bar{\xi})b(\bar{x},\xi)e^{(\bar{\bar{x}}-\bar{x})\cdot(\bar{\xi}-\xi)}d\bar{x}d\bar{\xi}\Big\}e^{(\bar{\bar{x}}-x)\cdot\xi}u(x)dxd\xi \end{aligned}$$

ここで、$\Big\{\cdots\Big\}$ のところが $a*b$ に相当しているからそこだけ取り出して計算すれば、

$$\begin{aligned} a*b &\sim \iint\Big(\sum\frac{1}{\alpha!}(\bar{\xi}-\xi)^\alpha\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}a(\bar{\bar{x}},\xi)\Big)b(\bar{x},\xi)c^{(\bar{\bar{x}}-\bar{x})\cdot(\bar{\xi}-\xi)}d\bar{x}d\bar{\xi} \quad \text{(Taylor 展開)}\\ &=\iint\Big(\sum\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}a(\bar{\bar{x}},\xi)\Big)b(\bar{x},\xi)\Big(\frac{-\hbar}{\sqrt{-1}}\Big)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial\bar{x}^\alpha}e^{(\bar{\bar{x}}-\bar{x})\cdot(\bar{\xi}-\xi)}d\bar{x}d\bar{\xi} \quad \text{(}e\text{の肩から)}\\ &=\iint\Big(\sum\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}a(\bar{\bar{x}},\xi)\Big)\Big(\frac{\hbar}{\sqrt{-1}}\Big)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial\bar{x}^\alpha}b(\bar{x},\xi)e^{(\bar{\bar{x}}-\bar{x})\cdot(\bar{\xi}-\xi)}d\bar{x}d\bar{\xi} \quad \text{(部分積分)}\\ &=\sum\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}a(\bar{\bar{x}},\xi)\Big(\frac{-\hbar}{\sqrt{-1}}\Big)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial\bar{x}^\alpha}b(\bar{\bar{x}},\xi) \quad \text{(反転公式)} \end{aligned}$$

このような積公式は OIT にもあるがかなり複雑である。大雑把に書けば、

$$(5.3)\qquad F(a,f,\tilde{\varphi})\circ F(b,g,\tilde{\psi}) = F(c(a,b,f,g,\tilde{\varphi},\tilde{\psi}),h(f,g,\tilde{\varphi},\tilde{\psi}),\tilde{\varphi}\circ\tilde{\psi})$$

となる。

（注意）本稿では、$(a,f,\tilde{\varphi})$ 等を局所的座標と考えることで $F(a,f,\tilde{\varphi})$ を要素とする集合に無限次元 Lie 群としての構造を導入しようとしているのである。

## 6 関数空間の設定.

いままでは、関数空間について何も言ってこなかったが、以降でもちいられる空間をここで設定しておく。

$C^\infty_{\mathbb{K}}(S^{2n-1})$ を $S^{2n-1}$ 上の全ての $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$)-valued $C^\infty$ 関数のなす空間、 そして $H^{-i}_{\mathbb{K}}$ を $\mathbb{R}^{2n}-\{0\}$ 上の 全ての homogeneous functions of degree $-i$ ( $i\in\mathbb{Z}$ ) のなす空間とする。変数 $\rho$ と $\omega$ を用いることによって、$H^{-i}_{\mathbb{K}}$ と $C^\infty_{\mathbb{K}}(S^{2n-1})\,\rho^{-i}$ とを同一視することにし、$H^{-i}_{\mathbb{K}}$ を ($C^\infty_{\mathbb{K}}(S^{2n-1})$ 上の $C^\infty$-topology) によって Fréchet space と考える。更に、$H^{-i}_{\mathbb{K}}$ にも $C^\infty_{\mathbb{K}}(S^{2n-1})$ から Fréchet topology を誘導しておく。

次に、関数空間 $\Sigma^{-m}_{\mathbb{K}}$ を

$$f(x,\xi)=f_{-m}(\omega)\rho^{-m}+f_{-m-1}(\omega)\rho^{-m-1}+\cdots f_{-m-i}(\omega)\rho^{-m-i}+\cdots\cdots, \tag{6.1}$$

なる漸近展開を持つような $C^\infty$- 関数全体とする。ここで $(x,\xi)=\rho\omega$.

$H^{-i}_{\mathbb{K}}$ の位相を用いることにより、$\Sigma^{-m}_{\mathbb{K}}$ に projective limit topology を誘導することができる。以下では、それについてもう少し詳しく見ることにする。

そのために任意の非負整数 $l$ にたいして、 関数空間 $\mathfrak{B}^{-l}_{\mathbb{K}}$ 以下のように定める。

$$\mathfrak{B}^{-l}_{\mathbb{K}}=\{r\in C^\infty_{\mathbb{K}}(\mathbb{R}^{2n})|\ \forall\alpha,\beta,\exists C_{\alpha,\beta};\langle x;\xi\rangle^l|\partial^\alpha_x\partial^\beta_\xi r(x,\xi)|\le C_{\alpha,\beta}\}. \tag{6.2}$$

この関数空間は 以下のようなノルム系 で Fréchet space になっている。

$$\|r\|_{-l,k}=\sum_{|\alpha|+|\beta|\le k}\sup_{(x,\xi)}\langle x;\xi\rangle^l|\partial^\alpha_x\partial^\beta_\xi r(x,\xi)|. \tag{6.3}$$

次に、

$$\Sigma^{-m,-l}_{\mathbb{K}}=H^{-m}_{\mathbb{K}}\oplus H^{-m-1}_{\mathbb{K}}\oplus\cdots\oplus H^{-l}_{\mathbb{K}}\oplus\mathfrak{B}^{-l-1}_{\mathbb{K}}\qquad(-l\le -m), \tag{6.4}$$

を Fréchet spaces の direct product space とする。ここで、inclusion mapping を

$$i^{-l-1}_{-l}:\Sigma^{-m,-l-1}_{\mathbb{K}}\to\Sigma^{-m,-l}_{\mathbb{K}} \tag{6.5}$$

ただし、

$$i^{-l-1}_{-l}\Big(\sum_{i=m}^{l+1}f_{-i}\rho^{-i}\oplus r_{-l-2}\Big)=\sum_{i=m}^{l}f_{-i}\rho^{-i}\oplus\{\kappa f_{-l-1}\rho^{-l-1}+r_{-l-2}\}. \tag{6.6}$$

とする。そうすると、$\{\Sigma^{-m,-l}_{\mathbb{K}},i^{-l-1}_{-l}\}$ は Fréchet spaces の projective system となる。そこで、$\Sigma^{-m}_{\mathbb{K}}$ によって projective limit $\varprojlim_{l}\Sigma^{-m,-l}_{\mathbb{K}}$ を表すことにする。

## 7 考察及び定理.

いろいろ考えると OIT の phase や symbol のなす関数空間に Fréchet topology をいれ、それを model space とする Fréchet-Lie 群 を考えればよさそうであることが分かる ([F1-2],[Ki],[KK] 参照) 。しかし、すべての invertible OIT などを考えたのでは regularity をもつかどうかが分

からない。そこで、 phase function が identity にちかい $S^{2n-1}$ 上の contact diffeomorphism $\tilde{\varphi}$ に対応する homogeneous degree 2 の real-valued function $S_{\tilde{\varphi}}$ と homogeneous degree 1 の real-valued function $f$ との和であり、 symbol が homogeneous degree 0 以下の complex valued function であるような OIT

$$\left(F(a,f,\tilde{\varphi})u\right)(x) = \iint a(\bar{x},\xi)e^{\frac{\sqrt{-1}}{\hbar}\{S_{\tilde{\varphi}}(\bar{x},\xi)+f(\bar{x},\xi)-\xi\cdot x\}}u(x)\bar{d}x\bar{d}\xi,$$

（ここで、 $\bar{d}x = (2\pi\hbar)^{-n/2}dx$ 等、）から生成される群 $GF^0$ を考えると regularity を持つことが分かり、次の定理を得る。

**Main theorem**(cf.[M1])

*Lie algebra* $\mathcal{G}$ として $\sqrt{-1}H^2_{\mathbb{R}} \oplus \sqrt{-1}H^1_{\mathbb{R}} \oplus \Sigma^0_{\mathbb{C}}$、つまり

$$\Big\{\sqrt{-1}a_2(\omega)\rho^2 + \sqrt{-1}a_1(\omega)\rho^1$$
$$+ a_0(\omega) + a_{-1}(\omega)\rho^{-1} + a_{-2}(\omega)\rho^{-2} + \cdots + a_{-m}(\omega)\rho^{-m} + \cdots$$

と漸近展開される $C^\infty$*-function* を *symbol* とする $\Psi DO$ 全体。

但し $\omega \in S^{2n-1}$, $\rho = \sqrt{x_1^2+\cdots+x_n^2+\xi_1^2+\cdots+\xi_n^2}$, $(x_1,\cdots,x_n,\xi_1,\cdots,\xi_n) = \rho\omega$

$$a_2, a_1 \in C^\infty_{\mathbb{R}}(S^{2n-1}), \quad a_{-i} \in C^\infty_{\mathbb{C}}(S^{2n-1})\ (i=0,1,2,\cdots)\Big\}$$

を持つ *RF-Lie* 群 $GF^0$ が *invertible inhomogeneous OIT* の *subgroup* として存在する。

ここで $a$ を *symbol* とする $\Psi DO$ とは

$$(P(a)u)(\bar{x}) = \iint a(\bar{x},\xi)e^{(\bar{x}-x)\cdot\xi}u(x)\bar{d}x\bar{d}\xi$$

なる作用素のことである ([*Ku*] 参照)。 $f, g \in \mathcal{G}$ にたいして *Lie bracket* は

$$[f,g] = \sum_{|\alpha|\geq 1}\left(\frac{\hbar}{\sqrt{-1}}\right)^{|\alpha|}\left\{\frac{\partial^{|\alpha|}f}{\partial\xi^\alpha}\frac{\partial^{|\alpha|}g}{\partial x^\alpha} - \frac{\partial^{|\alpha|}g}{\partial\xi^\alpha}\frac{\partial^{|\alpha|}f}{\partial x^\alpha}\right\}$$

となる。（*Dirac* の正準量子化の手続き $\{\cdot,\cdot\} \to \left(\frac{\sqrt{-1}}{\hbar}\right)[\cdot,\cdot]$ を思い出すべきである。） □

この $GF^0$ 上では、作用素 $\Phi_t$ に関する Schrödinger equation

$$\frac{d}{dt}\Phi_t = \sqrt{-1}\hat{H}_t\cdot\Phi_t, \quad \Phi_0' = Id. \qquad (\sqrt{-1}\hat{H}_t \in \mathcal{G})$$

が time-dependent vector field の integral curve として well-defined となり、 その解が product integral $\prod e^{\sqrt{-}\hat{H}_t dt}$ で構成される。ここで $\mathcal{G}$ の元の漸近展開の最高次数が 2 次のために $\hat{H}_t$ として 2 階の微分作用素 (調和振動子の Hamiltonian 等) が許される。以上で条件 (R) が確かめられたことになるので、次の節からは (#) について考える。

## 8 The modified wave front set.

波動方程式の研究に FIO と wave front set が用いられていることは良く知られている ([Hö] 参照)。このような考え方で、 Oscillatory integral transformation から classical part を取り出せるかどうかについて考えよう。そのために wave front set について復習しておく。

wave front set ([Hö], [Du], [Ku]) は Fourier integral operator についてのみ定義されている概念ではなく、一般の distribution $u$ にたいして、以下のように定義されている。

**Definition 8.1** *cotangent bundle* $T^*N$ の元 $\xi_x$ について或 $x$ の近傍上の *cut off function* $\phi$ があって $\hat{\phi u}(\tau\xi) = o(|\tau|^{-k}) \quad \forall k \in \mathbb{N}$ を満たすとき $\xi_x \notin WF(FIO(a,\phi))$ とする。

特に、Fourier integral operator

$$FIO(a,\phi) = \text{os-}\iint a(\bar{x},\theta,x)e^{\frac{\sqrt{-1}}{\hbar}\phi(\bar{x},\theta,x)}u(x)\bar{d}x\bar{d}\theta, \tag{8.1}$$

（但し $\phi(\bar{x},\theta,x)$ は $\theta$ について homogeneous degree 1 であり、この条件が oscillatory integral transformation のときとは異なるところである）における wave front set と $\phi$ の canonical graph $\varphi_\phi$(cf.§4) との間には密接な関係がある。大雑把に云って（本当は $a$ の support にも関係するのであるが、）

$$WF(FIO(a,\phi)) \subset \varphi_\phi \tag{8.2}$$

となっている。

そこで、この考えを我々の場合に適用したいのであるが、Schrödinger 方程式に現れる OIT は phase function が fiber 方向（運動量方向）の斉次性を持っていないために このままでは、canonical graph と wave front set との関係がうまくいかない。そこで Schrödinger 方程式にうまく馴染むように (singularity つまり canonical graph がとり出せるように)wave front set の定義を少し変形することを考えよう。そのために、wave front set の定義を少し別の角度から眺めてみる必要があるのだが、実は wave front set は次のように特徴付けられることが知られている。

$$WF(u) = \bigcap_{\substack{P(\tilde{a})u\in C^\infty\\ \tilde{a}\in S^0}} \gamma(\tilde{a}), \tag{8.3}$$

ここで $S^0$ は pseudo-differential operators の symbol class であり

$$\gamma(\tilde{a}) = \Big\{(x,\xi) \in T^*N - \{0\} : \underline{\lim}_{\tau\to\infty}|\tilde{a}(x,\tau\xi)| = 0\Big\} \tag{8.4}$$

(cf. [Ku]) である。

この式の右辺は、$P(\tilde{a})u$ が $C^\infty$ となるときに symbol $\tilde{a}$ の消えていなければならない方向を取り出そうとしているのである。そして実はそれが wave front set に一致しているというのが上の等式の主張なのである。

この定義では fiber 方向の無限遠方の状態のみ調べようとしているのであるが、それをやめて $(x,\xi)$ を平等に扱った定義に変形してみる (cf.[M2])。

**Definition 8.2**

$$\mathfrak{W}(I(a,\phi)) = \bigcap_{\substack{P(\tilde{a})I(a,\phi)\in\mathcal{S},\\ \tilde{a}\in H^0_{\mathbb{C}}(\mathbb{R}^{4n})}} \tilde{\gamma}(\tilde{a}), \tag{8.5}$$

ここで

$$\tilde{\gamma}(\tilde{a}) = \Big\{(\bar{x},\bar{\xi},x,-\xi) \in \mathbb{R}^{4n} : \lim_{t\to 0}|\tilde{a}(t\bar{x},t\bar{\xi},tx,t\xi)| = 0\Big\} \tag{8.6}$$

であり、$H^0_{\mathbb{C}}(\mathbb{R}^{4n})$ は $\mathbb{R}^{4n}$ 上或コンパクト近傍の外側で homogeneous degree 1 次の関数に一致する複素数値関数全体である。また、(8.3) と比較して $p(\tilde{a})I(a,\phi)$ の属する関数空間が急減少関数の空間 $\mathcal{S}$ であることも注意。このように定義された集合を modified wave front set とよぶことにする。

## 9 Contact diffeomorphism との関係.

この節では、前節で定義された modified wave front set と canonical diffeomorphism、contact diffeomorphism との関係を調べるのであるが、結論を云ってしまえば phase が 2 次と 1 次との和で書け、symbol にも或る仮定（感じとしては principal symbol が消えていないと言う条件）をつければ

$$\mathfrak{W}(I(a,\phi)) = \chi_{\phi_2} \tag{9.1}$$

が成立するのである。（$\chi_\phi$ ではなくて $\chi_{\phi_2}$ であることに注意。）

まず $\mathfrak{W}(I(a,\phi))$ は $\mathbb{R}^{4n}$ における集合であるが $\mathbb{R}^{2n}$ における symplectic transformation も $\mathbb{R}^{4n}$ における canonical graph と見なせることを注意しておく。

つぎに canonical graph と generating function との関係をみる。

$(\bar{x}_0, \bar{\xi}_0, \xi_0, -x_0) \notin \chi_{\phi_2}$ であるための必要かつ十分条件は、任意の $\theta$ に対して

(9.2) $(d_\theta\phi)(\bar{x},\theta,x) \neq 0$ または、$(d_{\bar{x}}\phi)(\bar{x},\theta,x) - \bar{\xi} \neq 0$ または、$(d_x\phi)(\bar{x},\theta,x) - \xi \neq 0.$

が成立することであるから、canonical graph の外側では

(9.3)
$$L = \frac{(d_\theta\phi)(\bar{x},\theta,x)\cdot\frac{\hbar}{\sqrt{-1}}\partial_\theta + \{(d_{\bar{x}}\phi)(\bar{x},\theta,x)-\bar{\xi}\}\cdot\frac{\hbar}{\sqrt{-1}}\partial_{\bar{x}} + \{(d_x\phi)(\bar{x},\theta,x)-\xi\}\cdot\frac{\hbar}{\sqrt{-1}}\partial_x}{\|\ (d_\theta\phi)(\bar{x},\theta,x), (d_{\bar{x}}\phi)(\bar{x},\theta,x)-\bar{\xi}, (d_x\phi)(\bar{x},\theta,x)-\xi\ \|^2}.$$

が well-defined（分母が 0 でない）となり

$$L\Big(e^{\frac{\sqrt{-1}}{\hbar}\{\phi(\bar{x}\theta,x)+\langle(\bar{z},z)-(\bar{x},x)|(\bar{\xi}-\xi)\rangle\}}\Big) = e^{\frac{\sqrt{-1}}{\hbar}\{\phi(\bar{x}\theta,x)+\langle(\bar{z},z)-(\bar{x},x)|(\bar{\xi}-\xi)\rangle\}} \tag{9.4}$$

をみたす。この微分作用素を用いると Lax-technique (部分積分によって $(L^*)^k$ を symbol に作用させて symbol の degree をさげる。[AF], [F1], [Hö], [Du], [Ku] を参照のこと。）がつかえて $P(\tilde{a})I(a,\phi) \in \mathcal{S}$ が示せる。

逆に、$\tilde{a}$ の support が $\phi$ の canonical graph と共通の元を持つときには stationary phase method(cf.[Hö]) によって $\mathbb{R}^{4n}$ における canonical graph の方向が決して消えることがないということが示せるので $P(\tilde{a})I(a,\phi) \notin \mathcal{S}$ が証明される。

以上見てきたように modified wave front set の定義式は canonical graph（更には対応する contact diffeomorphism) を選び出す操作 ($\pi$ と書く) とみなすことができる。

古典論的な群である $\mathcal{D}_\omega(S^{2N-1})$ と量子論的な群である $GF^0$ との関係は

$$1 \to GE^0 \to GF^0 \xrightarrow{\pi} \mathcal{D}_\omega(S^{2n-1})_0 \to 1$$

で与えられる。これが（#）に相当している。但し、ここで $\pi$ は OST にその modified wave front set を対応させる写像である。また、$GE^0$ は

$$\Big(F(a,f,\tilde{\varphi})u\Big)(x) = \iint a(\bar{x},\xi)e^{\frac{\sqrt{-1}}{\hbar}\{S_{\tilde{\varphi}}(\bar{x},\xi)+f(\bar{x},\xi)-\xi\cdot x\}}u(x)dxd\xi,$$

から生成される RF-Lie 群 である。

## 10 Proof of the Main Theorem.

最後に main theorem の証明の sketch をあたえておこう (cf.[M1])。証明は§9 の short exact sequence において $GE^0$ と $\mathcal{D}_\omega(S^{2n-1})_0$ が RF-Lie 群となることを示し、はさみうちの原理 (両脇が RF-Lie 群 なら真中もそう。cf.[OMYK]) によって $GF^0$ の regularity を示す。ところが $GE^0$ の regularity も以下に見るように別の short exact sequences

$$\begin{array}{ccccccccc}
1 & \to & G\Sigma^0 & \to & GE^0 & \to & H^1_{\mathbb{R}} & \to & 1 \\
1 & \to & G\Sigma^{-1} & \to & G\Sigma^0 & \to & H^0_{\mathbb{C},*} & \to & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & \to & G\Sigma^{-n} & \to & G\Sigma^{-n+1} & \to & H^{-n+1}_{\mathbb{C}} & \to & 1.
\end{array}$$

を考えて十分小さな $-n$ について $G\Sigma^{-n}$ の regularity を直接示すことによって証明がなされる。($\|a * b\|_k \leq C_k\|a\|_k \cdot \|b\|_{\delta_k}$ という不等式を解析的に証明するのであるが、十分小さな $-n$ についてなら *symbol* が積分可能となり取り扱いが多少楽になる。あとは $*$-exponential を行列の時のように上の不等式及び完備性をもちいて定める。cf.[OMYK]) ここで $G\Sigma^{-i}$ は $-i$ 次以下の invertible ΨDO のなす Fréchet-Lie 群で $H^{-i}_{\mathbb{K}}$ は $-i$ 次の $\mathbb{K}$-valued function のなす可換 Fréchet-Lie 群ある。また $H^0_{\mathbb{C},*}$ は 0 次の $C^*$-valued function が、かけ算でなす可換 Fréchet-Lie 群である。

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# On smooth Sp(2,$\mathbb{R}$)-actions on $S^4$

Kazuo Mukōyama

## 0. Introduction

In [2], Asoh classified smooth SL(2,C)-actions on $S^3$ topologically and in [7], Uchida classified $SO_0(p,q)$-actions on $S^{p+q-1}$ for p, q $\geq$ 3 such that the restricted SO(p)$\times$SO(q)-actions are standard. Each of their actions is characterized by a pair ($\varphi$, f) satisfying certain conditions, where $\varphi$ is a one-parameter transformation group on $S^1$ and f : $S^1 \to P_1(\mathbb{R})$ is a smooth function. The pair is introduced by Asoh and is improved by Uchida. That is constructed mainly by using the two facts : first, the restricted maximal compact subgroup action has codimension 1 principal orbits and secondly, the fixed point set of the restricted principal isotropy subgroup action is diffeomorphic to $S^1$.

In this report, we shall study smooth Sp(2,$\mathbb{R}$)-actions on $S^4$ without fixed points. Sp(2,$\mathbb{R}$) is simple and contains U(2) as a maximal compact subgroup. Under the condition, the principal isotropy subgroup of the restricted U(2)-action is conjugate to a 1-torus T. The U(2)-action has a codimension 1 principal orbits, but the fixed point set of restricted T-action is diffeomorphic to $S^2$. That is a little different from [2] and [7]. Therefore, we construct a triple (S, $\varphi$, f), instead of the pair, satisfying the conditions defined in § 4, where S is diffeomorphic to $S^1$ in $S^2$, $\varphi$ is a one-parameter transformation group on S and f : S $\to P_1(\mathbb{R})$ is a smooth map .

## 1. Preliminaries

**1.1.** Sp(2,$\mathbb{R}$) and $\mathfrak{sp}$(2,$\mathbb{R}$). Let Sp(2,$\mathbb{R}$) be the real symplectic group of order 2 defined by

$$Sp(2,\mathbb{R}) = \{g \in M(4,\mathbb{R}) \mid gJ{}^tg = J\} \text{ for } J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

where M(4, $\mathbb{R}$) is the set of real 4$\times$4 matrices. Sp(2,$\mathbb{R}$) contains U(2) as a maximal compact subgroup , which is naturally embedded in SO(4) by $k=k_1+i\,k_2 \to \begin{pmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{pmatrix}$. Let $\mathfrak{sp}$(2,$\mathbb{R}$) be the Lie algebra of Sp(2,$\mathbb{R}$). Then A $\in \mathfrak{sp}$(2,$\mathbb{R}$) if and only if $AJ+J{}^tA = O$. Hence

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix},$$

where $A_i$ is $2\times2$ matrices and $A_2$, $A_3$ are symmetric. Hence $\mathfrak{sp}(2,\mathbb{R})$ is a 10-dimensional Lie algebra. We take the following as a basis:

$$E_1=\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right),\ E_2=\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right),\ E_3=\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right),\ E_4=\left(\begin{array}{cc|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right),$$

$$E_5=\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right),\ E_6=\left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right),\ E_7=\left(\begin{array}{cc|cc} 1 & & & \\ & & & \\ \hline & & -1 & \\ & & & \end{array}\right),\ E_8=\left(\begin{array}{cc|cc} & & 1 & \\ & & & \\ \hline 1 & & & \\ & & & \end{array}\right),$$

$$E_9=\left(\begin{array}{cc|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right),\ E_{10}=\left(\begin{array}{cc|cc} & & & \\ & & & -1 \\ \hline & & & \\ & -1 & & \end{array}\right).$$

Especially, the Lie algebra $\mathfrak{u}(2)$ of $U(2)$ is now given by

$$\mathfrak{u}(2) = < E_1 , E_2 , E_3 , E_4 >,$$

where < > denote a Lie subalgebra generated by the elements in the angle bracket.

**1.2.** The 5-dimensional standard representation of $Sp(2,\mathbb{R})$. We denote an inner product on $M(4,\mathbb{R})$ by

$$(X,Y) = \mathrm{trace}(X\,{}^tY) \quad \text{for } X,Y \in M(4,\mathbb{R})$$

and define an action of $Sp(2,\mathbb{R})$ on $M(4,\mathbb{R})$ by

$$(1.1) \qquad g\cdot X = gX\,{}^tg\,, \quad \text{for } g\in Sp(2,\mathbb{R})\,,\ X\in M(4,\mathbb{R})\,.$$

Let $M_{alt} = \{X \in M(4,\mathbb{R}) \mid {}^tX = -X\}$. Then $M_{alt}$ is an $Sp(2,\mathbb{R})$-invariant subspace of $M(4,\mathbb{R})$ and has a basis:

$$e_1=1/2\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right),\ e_2=1/2\left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right),\ e_3=1/2\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right),$$

$$e_4=1/2\left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right),\ e_5=1/2\left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right),\ e_6=1/2\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right).$$

Since $e_6 = 1/2\, J$, the space $\mathbf{R}^5 = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ is also $Sp(2,\mathbf{R})$-invariant. We call this $\mathbf{R}^5$ the 5-dimensional standard representation space of $Sp(2,\mathbf{R})$ and the action (1.1) the standard action of $Sp(2,\mathbf{R})$ on $\mathbf{R}^5$. Let $R_1 = \mathrm{span}\{e_1, e_2\}$ and $R_2 = \mathrm{span}\{e_3, e_4, e_5\}$. Then $\mathbf{R}^5 = R_1 \oplus R_2$ and has following properties:

(1.2) The standard $Sp(2,\mathbf{R})$-action on $\mathbf{R}^5$ leaves invariant the quadratic form

$$-v_1^2 - v_2^2 + w_1^2 + w_2^2 + w_3^2,$$

for any element $X = v_1 e_1 + v_2 e_2 + w_1 e_3 + w_2 e_4 + w_3 e_5$ of $\mathbf{R}^5$.

(1.3) $R_1$ and $R_2$ are the $U(2)$-invariant subspaces. Moreover, $U(2)$ acts on $S(R_i)$ $(i=1, 2)$ transitively and

$$S(R_1) = U(2)/SU(2), \quad S(R_2) = U(2)/T^2,$$

where $S(R_i) = \{X \in R_i \mid \|X\| = 1\}$. Thus $U(1)$, a normal subgroup of $U(2)$, acts trivially on $R_2$ and $SU(2)$ acts trivially on $R_1$.

**1.3.** Subgroups and subalgebras: Put $\mathbf{R}^3 = \mathrm{span}\{e_1, e_2, e_3\} \subset \mathbf{R}^5$. Let $H(a,b,c)$ (resp. $\mathfrak{h}(a,b,c)$) denote the isotropy subgroup of $Sp(2,\mathbf{R})$ (resp. the isotropy subalgebra of $\mathfrak{sp}(2,\mathbf{R})$) at $X = ae_1 + be_2 + ce_3$ for $(a,b,c) \neq (0,0,0)$. Then $A \in \mathfrak{h}(a, b, c)$ if and only if $AX + X^tA = O$. Hence we have

**Fact 1.4.** $\mathfrak{h}(a,b,c) = \langle B_1, B_2, B_3, B_4, B_5, B_6 \rangle$,

where in case of $c \neq 0$,

$B_1 = bE_3 + aE_4 + 2cE_7$, $B_2 = -aE_3 + bE_4 + 2cE_8$, $B_3 = -bE_3 + aE_4 + 2cE_9$,

$B_4 = aE_3 + bE_4 - 2cE_{10}$, $B_5 = E_2$, $B_6 = -c(E_1 - E_2) + aE_5 + bE_6$,

and in case of $c = 0$,

$B_1 = r^2E_7 + (b^2 - a^2)E_9 + 2abE_{10}$, $B_2 = r^2E_8 - 2abE_9 + (b^2 - a^2)E_{10}$, $B_3 = E_3$,

$B_4 = E_4$, $B_5 = E_2$, $B_6 = aE_5 + bE_6$,

for $a^2 + b^2 = r^2 \neq 0$.

We define $m(\theta) \in Sp(2,\mathbf{R})$ by

$$(1.5) \qquad m(\theta) = \exp(-\theta/2\, E_5) = \left(\begin{array}{cc|cc} \cosh\theta/2 & -\sinh\theta/2 & & \\ -\sinh\theta/2 & \cosh\theta/2 & \multicolumn{2}{c}{0} \\ \hline & & \cosh\theta/2 & \sinh\theta/2 \\ \multicolumn{2}{c|}{0} & \sinh\theta/2 & \cosh\theta/2 \end{array}\right),$$

and put $M = \{ m(\theta) \mid \theta \in \mathbb{R} \}$. Then we have

(1.6) $$m(\theta) \cdot (a\, e_1 + b\, e_2 + c\, e_3) = a\, e_1 + b'\, e_2 + c'\, e_3,$$

where $b' = b \cosh\theta + c \sinh\theta$, $c' = b \sinh\theta + c \cosh\theta$.

Let $T = \{ \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \in SU(2) \mid |t| = 1 \}$ be a maximal torus of $SU(2)$. Then we have

$$\mathfrak{t} = \langle E_2 \rangle,\ \mathrm{Lie}(N(T, Sp(2,\mathbb{R}))) = \langle E_1, E_2, E_5, E_6 \rangle,$$

where $\mathfrak{t}$ and $\mathrm{Lie}(N(T, Sp(2,\mathbb{R})))$ are the Lie algebras of $T$ and $N(T, Sp(2,\mathbb{R}))$, respectively and $N(T, Sp(2,\mathbb{R}))$ is the normalizer of $T$ in $Sp(2,\mathbb{R})$. By (1.2), (1.3), we have the following.

**Fact 1.7.** $Sp(2,\mathbb{R}) = U(2)MH(0,b,c)$ for $(0,b,c) \neq (0, 0, 0)$.

By Fact 1.4, it is clear that

$$\bigcap_{(a,b,c)} \mathfrak{h}(a,b,c) = \mathfrak{t},$$

where the intersection is taken over all $(a,b,c) \neq (0,0,0)$.

**Lemma 1.8.** Let $\mathfrak{g}$ be a proper subalgebra of $\mathfrak{sp}(2,\mathbb{R})$ which contains $\mathfrak{t}$. If $\dim \mathfrak{g} \geq 6$, then $\mathfrak{g} = \mathfrak{h}(a, b, c)$ for some $(a, b, c) \neq (0, 0, 0)$ or $\mathfrak{g} = \mathfrak{h}(a, b, c) \oplus \theta^1(a,b,c)$ for $a^2 + b^2 = c^2 \neq 0$, where the one-dimensional space $\theta^1(a,b,c)$ is generated by a matrix $bE_5 - aE_6$.

Proof. By considering the $Ad(T)$-action on $\mathfrak{sp}(2,\mathbb{R})$, we can first decompose $\mathfrak{sp}(2,\mathbb{R})$ into $Ad(T)$-invariant subspaces as vector spaces:

$$\mathfrak{sp}(2,\mathbb{R}) = V_1 \oplus V_2 \oplus V_3 \oplus W,$$

where $V_1 = \mathrm{span}\{ E_3, E_4 \}$, $V_2 = \mathrm{span}\{ E_7, E_8 \}$, $V_3 = \mathrm{span}\{ E_9, E_{10} \}$, $W = \mathrm{span}\{ E_1, E_2, E_5, E_6 \}$, and $Ad(T)$ acts on $W$ trivially. By considering the $Ad(T)$-action on $V_i$, we have

$$\mathfrak{g} = \mathfrak{g} \cap (V_1 \oplus V_2 \oplus V_3) \oplus \mathfrak{g} \cap W.$$

Next, we obtain the result by considering the Lie algebra structure of $\mathfrak{sp}(2,\mathbb{R})$ and the bracket operations on these $Ad(T)$-invariant subspaces (cf. Uchida[6], § 2).

q.e.d.

By Fact 1.4, we see that $\mathfrak{h}(a,b,c) = \mathfrak{h}(a',b',c')$ if and only if $(a,b,c) = r\,(a',b',c')$ for $0 \neq r \in \mathbb{R}$. For this reason, hereafter we rewrite $H(a,b,c)$ (resp. $\mathfrak{h}(a,b,c)$) as $H(a{:}b{:}c)$ (resp. $\mathfrak{h}(a{:}b{:}c)$), where $(a{:}b{:}c)$ is an element of the real projective space $P_2(\mathbb{R})$.

Next we denote the element $t(\tau) \in U(2)$ by

$$t(\tau) = \exp(-\tau/2\, E_1) = \left(\begin{array}{cc|cc} \cos\tau/2 & & -\sin\tau/2 & \\ & \cos\tau/2 & & -\sin\tau/2 \\ \hline \sin\tau/2 & & \cos\tau/2 & \\ & \sin\tau/2 & & \cos\tau/2 \end{array}\right) \text{ for } \tau \in \mathbf{R}.$$

Then $\{t(\tau) \mid \tau \in \mathbf{R}\} = U(1)$ is a normal subgroup of $U(2)$ and acts on $\mathbf{R}^3$ by

(1.9) $\qquad t(\tau)\cdot(a\,e_1+b\,e_2+c\,e_3) = a'\,e_1+b'\,e_2+c\,e_3,$

where $a' = a\cos\tau - b\sin\tau$, $b' = a\sin\tau + b\cos\tau$. The M- and U(1)-actions on $\mathbf{R}^3$, which are the restrictions of the standard Sp(2,$\mathbf{R}$)-action on $\mathbf{R}^5$, derive M- and U(1)-actions on $P_2(\mathbf{R})$, respectively. We call these derived actions on $P_2(\mathbf{R})$ the standard actions on $P_2(\mathbf{R})$ and use the same notation as the actions on $\mathbf{R}^3$.

## 2. Standard Sp(2,R)-action on $S^4$

We set $S^4 = \{ X \in \mathbf{R}^5 \mid \|X\| = 1 \}$. Let $\Phi_o$: Sp(2,$\mathbf{R}$)$\times S^4 \to S^4$ denote the smooth Sp(2,$\mathbf{R}$)-action on $S^4$ defined by

(2.1) $\qquad \Phi_o(g, X) = \|g\cdot X\|^{-1}\, g\cdot X.$

We call the action (2.1) the standard action of Sp(2,$\mathbf{R}$) on $S^4$. By (1.3), (1.5), this action has next properties:

(2.2) The restricted U(2)-action $\psi$ has the principal orbit U(2)/T of codimension 1 and two singular orbits $U(2)/T^2$ and U(2)/SU(2). Let F(T) be the fixed point set of the restricted T-action on $S^4$. Then $F(T) = \{u\,e_1 + v\,e_2 + w\,e_3 \mid u^2+v^2+w^2 = 1\} \subset \mathbf{R}^3$ and

$$F(T)/(N(T,U(2))/T) = S^4/U(2),$$

where $N(T,U(2)) = T^2 \cup N_o T^2$ for $N_o = \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right)$ (cf. Bredon[3], p. 191).

(2.3) $\quad S = \{v\,e_2 + w\,e_3 \mid v^2 + w^2 = 1\}$ is an M-invariant subspace of F(T).

By (1.6), Fact 1.7, (1.9), (2.2), (2.3), we see that the standard Sp(2,$\mathbf{R}$)-action on $S^4$ consists of three orbits.

**Remark 2.4.** By the classification theorem due to Asoh [1], any almost effective smooth U(2)-action on $S^4$ is equivariantly diffeomorphic to one of the following:

(1) the $U(2)$-action $\psi$ defined above.

(2) $\psi' : U(2) \times S^4 \to S^4$ defined by

$$\psi'(g, (x, y)) = (g\,x, y) \text{ for } (x, y) \in S^4 \subset C^2 \times \mathbb{R}^1.$$

We notice that the action $\psi'$ has two fixed points as singular orbits.

### 3. Smooth Sp(2,R)-actions on $S^4$

**Fact 3.1.** Let $\Phi : Sp(2,\mathbb{R}) \times N \to N$ be a smooth $Sp(2,\mathbb{R})$-action on a smooth 4-manifold N. Then, the action has fixed points if and only if the restricted $U(2)$-action has fixed points.

This fact follows from Lemma 1.8. By this fact and Remark 2.4, we have

**Fact 3.2.** Let $\Phi : Sp(2,\mathbb{R}) \times S^4 \to S^4$ be a smooth $Sp(2,\mathbb{R})$-action on $S^4$. Then the action has no fixed points if and only if the restricted $U(2)$-action is equivariantly diffeomorphic to the action $\psi$ defined above.

Hereafter we shall consider smooth $Sp(2,\mathbb{R})$-actions on $S^4$ without fixed points. By Fact 3.2, we assume that the restricted $U(2)$-action is coincides with $\psi$. We put

(3.3)
$$G = Sp(2,\mathbb{R}),\ K = U(2),\ T = \left\{ \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \mid |t| = 1 \right\},$$
$$\psi = \Phi_0 \mid K \times S^4,\ F(T) = \{ue_1 + ve_2 + we_3 \mid u^2 + v^2 + w^2 = 1\}.$$

We shall identify $F(T)$ with the 2-sphere $S^2$ by the natural diffeomorphism $h : S^2 \to F(T)$ defined by $h(u,v,w) = ue_1 + ve_2 + we_3$.

Let $\Phi : G \times S^4 \to S^4$ be a smooth G-action on $S^4$ satisfying $\Phi \mid K \times S^4 = \psi$. We shall construct a smooth map $f : F(T) \to P_2(\mathbb{R})$ uniquely determined by the condition

(3.4) $\qquad \mathfrak{h}(f(X)) \subset \mathfrak{g}_X \quad$ for $X \in F(T)$,

where $\mathfrak{g}_X$ is the isotropy subalgebra at X with respect to the given G-action $\Phi$ and $\mathfrak{h}(f(X))$ is a subalgebra of $\mathfrak{sp}(2,\mathbb{R})$ described in Fact 1.4. Because $\mathfrak{g}_X$ is a proper subalgebra of $\mathfrak{sp}(2,\mathbb{R})$ which contains $\mathfrak{t}$, there exists uniquely $(a:b:c) \in P_2(\mathbb{R})$ such that

(3.5) $\qquad \mathfrak{h}(a:b:c) \subset \mathfrak{g}_X$

by Lemma 1.8. By Lemma 1.8 and the fact that each element of $\mathfrak{sp}(2,\mathbb{R})$ can be considered naturally as a smooth vector field on $S^4$ (cf. Palais[5], ch. II, Th. II), we have

**Fact 3.6.** f is smooth and surjective.

Comparing $\mathfrak{h}(a:b:c)$ with the isotropy subalgebra of the restricted K-action , we have the following (3.7) and (3.8):

(3.7) $\quad f(X) = (0:0:1) \Leftrightarrow X = \pm\, \mathbf{e}_3$.

(3.8) Put $f(X) = (a:b:c)$ for $X = u\,\mathbf{e}_1 + v\,\mathbf{e}_2 + w\,\mathbf{e}_3$. Then

$$c = 0 \Leftrightarrow w = 0.$$

Let $m(\theta)$ be the matrix defined by (1.5). Then the set F(T) is invariant under the M-action $\Phi(m(\theta), -)$, because $m(\theta)$ commutes with each element of T. Let $\varphi : \mathbb{R}\times F(T) \to F(T)$ denote the smooth $\mathbb{R}$-action on F(T) defined by $\varphi(\theta, X) = \Phi(m(\theta), X)$. Then we see that f is U(1)- and M- equivariant by (1.6), (1.9) and the definitions of f and $\mathfrak{h}(a:b:c)$. Hence we have

(3.9) $\quad f(X) = (a:b:c) \Rightarrow f(\varphi(\theta, X)) = m(\theta)\cdot(a:b:c) = (a:b':c')$,

where $b' = b\cosh\theta + c\sinh\theta$ , $c' = b\sinh\theta + c\cosh\theta$ , and also have

(3.10) $\quad f(X) = (a:b:c) \Rightarrow f(t(\tau)\cdot X) = t(\tau)\cdot(a:b:c) = (a':b':c)$,

where $a' = a\cos\tau - b\sin\tau$, $b' = a\sin\tau + b\cos\tau$. Let $J_i : F(T) \to F(T)$ $(i = 1, 2)$ denote involutions defined by $J_1(u,v,w) = (-u, -v, w)$ and $J_2(u, v, w) = (u, v, -w)$. Then $J_1J_2(X) = -X$ and we have

(3.11) $\quad f(X) = (a:b:c) \Rightarrow f(J_1(X)) = f(J_2(X)) = (a:b:-c)$.

This follows from the fact $J_i(X) = \psi(j_i, X)$ $(i = 1, 2)$, where

(3.12) $$j_1 = \left(\begin{array}{c|c} & I_2 \\ \hline -I_2 & \end{array}\right) \in U(1),\quad j_2 = \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right) \in N(T, U(2)).$$

Since $j_i m(\theta) = m(-\theta) j_i$, we have

(3.13) $\quad J_i(\varphi(\theta, X)) = \varphi(-\theta, J_i(X))$ $(i = 1, 2)$.

We define $P_1(\mathbb{R}) = \{(a:b:c) \in P_2(\mathbb{R}) \mid a = 0\}$ and $S = f^{-1}(P_1(\mathbb{R}))$. Then we have

**Fact 3.14.** S is a 1-dimensional submanifold of F(T) which is diffeomorphic to a great circle in F(T).

Now we denote the restriction of f and $\varphi$ on S also by f and $\varphi$, respectively. Then S is $J_i$-invariant $(i = 1, 2)$ and these f and $\varphi$ also satisfy the conditions (3.7), (3.8), (3.9), (3.11) and (3.13). Moreover $S - \{\pm\mathbf{e}_3\}$ intersects U(1)-orbits on $F(T) - \{\pm\mathbf{e}_3\}$ transversely.

## 4. Properties of $(S, \varphi, f)$

Let $S^2 = \{ X = (u,v,w) \mid u^2+v^2+w^2 = 1 \}$ and $P_1(\mathbb{R}) = \{(0:b:c)\} \subset P_2(\mathbb{R})$. Let $(S, \varphi, f)$ be a triple of a 1-dimensional closed submanifold S, a smooth $\mathbb{R}$-action $\varphi : \mathbb{R}\times S \to S$ and a smooth map $f : S \to P_1(\mathbb{R})$ satisfying the following conditions :

(i) S is diffeomorphic to a great circle of $S^2$ and $S-\{(0,0, \pm 1)\}$ intersect circles $\{(u,v,w) \in S^2 \mid w = c\}$ $(-1 < c < 1)$ transversely.

(ii) $J_i(\varphi(\theta, X)) = \varphi(-\theta, J_i(X))$,

(iii) $f(X) = (0:b:c) \Rightarrow f(J_1(X)) = f(J_2(X)) = (0:b:-c)$,

(iv) $f(X) = (0:b:c) \Rightarrow f(\varphi(\theta, X)) = (0:b':c')$,

where $b' = b \cosh \theta + c \sinh \theta$, $c' = b \sinh \theta + c \cosh \theta$,

(v) $f(X) = (0:0:1) \Leftrightarrow X = (0,0,\pm 1)$,

(vi) $f(X) = (0:1:0) \Leftrightarrow X = (u,v,0) \in S$ .

Let $(S, \varphi, f)$ be a triple defined above. Let $W_{bc}$, $P(X)$ denote matrices defined by

$$(4.1)\qquad \begin{aligned} W_{bc} &= (b^2 +c^2)^{-1/2}(be_2 + ce_3), \\ P(X) &= W_{bc}\, {}^tW_{bc} = (b^2 +c^2)^{-1}(be_2 + ce_3)\, {}^t(be_2 + ce_3) \end{aligned}$$

for $f(X) = (0:b:c)$, respectively. Let $U(X)$ denote the subgroup of G defined by

$$(4.2)\qquad U(X) = \{ g \in G \mid (g \cdot W_{bc})\,{}^t(g \cdot W_{bc}) = W_{bc}\, {}^tW_{bc} \}.$$

Then $\operatorname{trace}P(X) = 1$ and $H(0:b:c) \subset U(X)$. Since $m(\theta)W_{bc} = W_{bc}m(\theta)$, we have

$$(4.3)\qquad (m(\theta) \cdot W_{bc})\,{}^t(m(\theta) \cdot W_{bc}) = \lambda(\theta, X)P(\varphi(\theta, X)),$$

where

$$\lambda(\theta, X) = \cosh 2\theta + 2bc(b^2 + c^2)^{-1} \sinh 2\theta$$

for $f(X) = (0:b:c)$. By the conditions (v), (vi), we have

$$(4.4)\qquad K \cap H(0:b:c) = K_X,$$

where $K_X$ denote the isotropy subgroup at $X \in S$ with respect to the K-action $\psi$.

## 5. Construction of $Sp(2,\mathbb{R})$-actions

**5.1.** Let $(S, \varphi, f)$ be a triple of a 1-dimensional closed smooth submanifold S of $S^2$, a smooth $\mathbb{R}$-action $\varphi$ on S and a smooth map $f : S \to P_1(\mathbb{R})$ satisfying the six conditions in § 4. We shall show how to construct a smooth G-action on $S^4$ from the triple $(S, \varphi, f)$. We use the notations (3.3), (3.12).

Let $X \in S$. Then by Fact 1.7,

$$(5.1)\qquad G = KMH(0:b:c)$$

for $f(X) = (0{:}b{:}c)$. Take $(g, p) \in G \times S^4$. Let us choose

$$
(5.2) \qquad \begin{aligned} &k \in K,\ X \in S : \psi(k, X) = p, \\ &k' \in K,\ \theta \in \mathbb{R},\ u \in H(0{:}b{:}c) : gk = k'\, m(\theta)u, \end{aligned}
$$

and put

$$
(5.3) \qquad \Phi(g, p) = \psi(k', \varphi(\theta, X)) \in S^4.
$$

Then we have the following.

**Proposition 5.4.** $\Phi : G \times S^4 \to S^4$ of (5.3) is a smooth $G$-action on $S^4$ such that $\Phi \mid K \times S^4 = \psi$.

**5.2.** First we shall show that $\Phi$ of (5.3) is well-defined and an abstract $G$-action on $S^4$ such that $\Phi \mid K \times S^4 = \psi$.

**Fact 5.5.** Let $f(X) = (0{:}b{:}c)$ and let $km(\theta)u = k'm(\theta')u'$ for $k, k' \in K$ and $u, u' \in H(0{:}b{:}c)$. Then $\psi(k, \varphi(\theta, X)) = \psi(k', \varphi(\theta', X))$.

By using this fact, we show that $\Phi$ of (5.3) is well-defined and an abstract action. First we show that $\Phi$ is well-defined. For any $(g, p) \in G \times S^4$, let us choose as in (5.2);

$$
p = \psi(k_1, X_1) = \psi(k_2, X_2),
$$

$$
gk_i = k_i' m(\theta_i) u_i,\ u_i \in H(0{:}b_i{:}c_i),
$$

where $f(X_i) = (0{:}b_i{:}c_i)$. By the first equation and the definition of the $K$-action $\psi$, we have $X_1 = J_1^r J_2^s(X_2)$ for some $r, s \in \{1, 2\}$. Hence $k_2^{-1} k_1 j_1^r j_2^s \in K_{X_2} \subset H(0{:}b_2{:}c_2)$ by (4.4). Hence we may put $k_2 = k_1 j_1^r j_2^s u_2'$ for some $u_2' \in H(0{:}b_2{:}c_2)$. Then we have

$$
\begin{aligned} k_2' m(\theta_2) u_2 = gk_2 = gk_1 j_1^r j_2^s u_2' &= k_1' m(\theta_1) u_1 j_1^r j_2^s u_2' \\ &= k_1' j_1^r j_2^s m((-1)^{r+s}\theta_1)(j_1^r j_2^s u_1)(j_1^r j_2^s u_2'). \end{aligned}
$$

Put $k_2'' = k_1' j_1^r j_2^s$. Then $k_2'' \in K$. On the other hand,

$$
(0{:}b_1{:}c_1) = (0{:}b_2{:}(-1)^{r+s} c_2),
$$

because $X_1 = J_1^r J_2^s(X_2)$. Hence we see

$$
(j_1^r j_2^s u_1)(j_1^r j_2^s u_2') \in H(0{:}b_2{:}c_2),
$$

by using the standard action of $j_1^r j_2^s \in K$ on $\mathbb{R}^3$. Hence we have

$$
\begin{aligned} \psi(k_2', \varphi(\theta_2, X_2)) = \psi(k_2'', \varphi((-1)^{r+s}\theta_1, X_2)) &= \psi(k_1' j_1^r j_2^s, \varphi((-1)^{r+s}\theta_1, X_2)) \\ &= \psi(k_1', J_1^r J_2^s \varphi((-1)^{r+s}\theta_1, X_2)) = \psi(k_1', \varphi(\theta_1, X_1)) \end{aligned}
$$

by Fact 5.5 and the conditions (ii), (iii).

Secondly we show that $\Phi$ is an abstract action. Take g, g' $\in$ G and p $\in S^4$. Let us choose as in (5.2);

$$p = \psi(k, X),\ gk = k'm(\theta)u,\ u \in H(0{:}b{:}c),$$
$$g'k' = k''m(\theta')u',\ u' \in H(0{:}b'{:}c'),$$

where $f(X) = (0{:}b{:}c)$ and $f(\varphi(\theta, X)) = (0{:}b'{:}c')$. Then

$$g'gk = g'k'm(\theta)u = k''m(\theta')u'm(\theta)u = k''m(\theta+\theta')(m(-\theta)u')(m(\theta)u)$$

and $(m(-\theta)u')(m(\theta)u) \in H(0{:}b{:}c)$ by the condition (iv). Hence we have

$$\Phi(g', \Phi(g, X)) = \Phi(g', \psi(k', \varphi(\theta, X))) = \psi(k'', \varphi(\theta', \varphi(\theta, X)))$$
$$= \psi(k'', \varphi(\theta'+\theta, X)) = \Phi(g'g, p).$$

Finally, take $(k, p) \in K\times S^4$ and put $p = \psi(k', X)$ as in (5.2). Then

$$\Phi(k, p) = \psi(kk', \varphi(0, X)) = \psi(kk', X) = \psi(k, \psi(k', X)) = \psi(k, p).$$

Hence $\Phi \mid K\times S^4 = \psi$.

**5.3.** Next we shall show the smoothness of $\Phi$ of (5.3).

For i = 1, 2, define

$$S_i(\Phi) = \{\ \Phi(g, e_{i+1}) \mid g \in G\ \},\ S_i(\Phi_o) = \{\ \Phi_o(g, e_{i+1}) \mid g \in G\ \}$$

for the G-action $\Phi$ of (5.3) and the standard G-action $\Phi_o$, respectively. Then it is clear that

$$S_1(\Phi_o) = \{v \oplus w \in S(R_1 \oplus R_2) \mid \|v\| > \|w\|\ \},$$
$$S_2(\Phi_o) = \{v \oplus w \in S(R_1 \oplus R_2) \mid \|v\| < \|w\|\ \}.$$

By (5.3) and the conditions of (S, $\varphi$, f), there exist positive real numbers $r_i < 1$ (i = 1, 2 ) such that

$$S_1(\Phi) = \{v \oplus w \in S(R_1 \oplus R_2) \mid \|w\| < r_1\},$$
$$S_2(\Phi) = \{v \oplus w \in S(R_1 \oplus R_2) \mid \|v\| < r_2\}.$$

**Fact 5.6.** There exist G-equivariant diffeomorphisms $F_i' : S_i(\Phi) \to S_i(\Phi_o)$ (i=1,2).

Hence we see that the restrictions $\Phi \mid G\times S_i(\Phi)$ (i=1,2) are smooth.

Put $f(X) = (0{:}b{:}c)$ and $X = (u,v,w)$. If $w > 0$, then $c \neq 0$ by the conditions (v), (vi) and there is a smooth function $\beta$ on $\{(u,v,w) \in S \mid w > 0\}$ such that $f(X) = (0{:}\beta(X){:}1)$. We define $S_+$ and $D_+$ by

$$S_+ = \{\ X = (u,v,w) \in S \mid w > 0,\ \beta(X) > 0 \text{ for } f(X) = (0{:}\beta(X){:}1)\},$$
$$D_+ = \{(\theta, X) \in \mathbb{R}\times S_+ \mid \varphi(\theta, X) \in S_+\}.$$

Let $W_+$ be the set of pairs $(g, X) \in G\times S_+$ satisfying the following condition : $(\theta, X) \in D_+$ for a decomposition $g = km(\theta)u$, where $f(X) = (0{:}\beta(X){:}1)$, $k \in K$ and $u \in H(0{:}\beta(X){:}1)$.

We have the following facts

**Fact 5.7.** Let $(\theta, X) \in \mathbf{R}\times S_+$ be given. Then $\varphi(\theta, X) \in S_+$ if and only if

(5.8) $$\{2\, \beta(X) \cosh 2\theta + (1 + \beta(X)^2) \sinh 2\theta\} > 0\,.$$

**Fact 5.9.** Let $X \in S_+$ and $f(X) = (0{:}\beta(X){:}1)$. Then for $g \in G$, $(g, X) \in W_+$ if and only if

(5.10) $$\pm\, \mathrm{trace}(g \cdot W_{\beta(X)1})^t(g \cdot W_{\beta(X)1}) \neq (1 - \beta(X)^2)\,(1 + \beta(X)^2)^{-1},$$

where $W_{\beta(X)1}$ is the matrix defined in (4.1).

By Fact 1.7, for any $g \in G$ we always have a decomposition $g = km(\theta)u$, where $k \in K$, $\theta \in \mathbf{R}$ and $u \in H(0{:}\beta(X){:}1)$. Then we see that

(*) $$\mathrm{trace}(g \cdot W_{\beta(X)1})^t(g \cdot W_{\beta(X)1}) = \cosh 2\theta + 2\beta(X)(\beta(X)^2 + 1)^{-1}\sinh 2\theta$$

by (4.3).

**Lemma 5.11.** For any $(g, X) \in W_+$, there exist uniquely $kT \in K/T$ and $\theta \in \mathbf{R}$ such that

(5.12) $$g = km(\theta)u\ ;\ u \in H(0{:}\beta(X){:}1),\ (\theta, X) \in D_+,$$

where $f(X) = (0{:}\beta(X){:}1)$. Furthermore, the correspondence $\triangle : W_+ \to K/T\times D_+$ defined by $\triangle(g, X) = (kT, (\theta, X))$ is smooth.

Proof. First we shall show the uniqueness of the decomposition. If $g = km(\theta)u = k'm(\theta')u'$, then $\|m(\theta) \cdot (0, \beta(X), 1)\| = \|m(\theta') \cdot (0, \beta(X), 1)\|$. Hence we have

$$2\beta(X) \sinh 2\theta + (1 + \beta(X)^2)\cosh 2\theta = 2\beta(X) \sinh 2\theta' + (1 + \beta(X)^2)\cosh 2\theta'.$$

We denote the left side of this equation by $\alpha(\theta)$. Then $\alpha(\theta)$ is an increasing function by Fact 5.7. Hence $\theta = \theta'$. This derives $k^{-1}k' \in T$. Next we shall show that $\triangle$ is smooth. Let $\theta = \theta(g, X)$ and $\delta(g, X) = kT$. We denote the smooth function $\gamma$ on $W_+ \times \mathbf{R}$ defined by

$$\gamma(g, X, \theta) = \cosh 2\theta + 2\, \beta(X)(1 + \beta(X)^2)^{-1}\sinh 2\theta - \mathrm{trace}((g \cdot W_{\beta(X)1})^t(g \cdot W_{\beta(X)1})),$$

for $f(X) = (0{:}\ \beta(X){:}1)$. Then $\gamma(g, X, \theta(g, X)) = 0$ by (5.12) and (*) . Furthermore, if $\gamma(g, X, \theta) = 0$, then

$$\partial\gamma/\partial\theta = 2(\sinh 2\theta + 2\beta(X)(1 + \beta(X)^2)^{-1} \cosh 2\theta) > 0$$

by Fact 5.7. Thus we see that the function $\theta(g, X)$ is smooth by the implicit function theorem.

Next consider the smooth mappings $\delta_1 : W_+ \to \mathbf{R}^5$, $\delta_3 : K/T \to \mathbf{R}^5$ and the smooth map $\delta_2$ on $(R_1 - \{\mathbf{o}\}) \oplus (R_2 - \{\mathbf{o}\})$ defined by

$$\delta_1(g, X) = (1 + \beta(X)^2)^{-1/2}\, g \cdot (\beta(X)\mathbf{e}_2 + \mathbf{e}_3),$$

$\delta_3(kT) = k \cdot (e_2 + e_3)$,

$\delta_2(v \oplus w) = \|v\|^{-1} v \oplus \|w\|^{-1} w$,

respectively. Then we see that $\delta_3\ \delta = \delta_2\ \delta_1$. Hence $\delta$ is smooth, because $\delta_1$ and $\delta_2$ are smooth maps and $\delta_3$ is an embedding.

q.e.d.

Define $W(\Phi) = \{(g, \psi(k, X)) \mid (gk, X) \in W_+\}$. We see that $W(\Phi)$ is an open set of $G \times S^4$, since $W_+$ is an open set of $G \times S_+$ by Fact 5.9. Moreover, we see that $\Phi \mid W(\Phi)$ is smooth, because $\Delta$ is smooth by Lemma 5.11. Therefore, $\Phi$ is smooth on $G \times S^4$, because $G \times S^4$ is covered by three open sets $G \times \{\Phi(g, e_2) \mid g \in G\}$, $G \times \{\Phi(g, e_3) \mid g \in G\}$ and $W(\Phi)$, and $\Phi$ is smooth on each open set.

**6. Equivalences and Theorem**

Let $\Phi_i$ $(i = 1, 2)$ be smooth G-actions on $S^4$ without fixed points. We say $\Phi_1$ and $\Phi_2$ are equivalent if $\Phi_1$ is equivariantly diffeomorphic to $\Phi_2$, i. e., if there exists a diffeomorphism $\Psi$ : $S^4 \to S^4$ satisfying $\Psi(\Phi_1(g, X)) = \Phi_2(g, \Psi(X))$ for any $(g, X) \in G \times S^4$.

We say that triples $(S_i, \varphi_i, f_i)$ $(i = 1, 2)$ satisfying the conditions (i) to (vi) in § 4 are equivalent if there exists a diffeomorphism $\xi$ from $S_1$ onto $S_2$ such that $\xi J_j = J_j \xi$ for $j = 1, 2$ and the following diagram is commutative:

$$\begin{tikzcd} \mathbb{R} \times S_1 \arrow[r, "\varphi_1"] \arrow[d, "1 \times \xi"'] & S_1 \arrow[dr, "f_1"] \arrow[d, "\xi"] & \\ & & P_1(\mathbb{R}) \\ \mathbb{R} \times S_2 \arrow[r, "\varphi_2"] & S_2 \arrow[ur, "f_2"'] & \end{tikzcd} \tag{6.1}$$

We have the following facts.

**Fact 6.2.** Let $\Phi_i$ $(i = 1, 2)$ be smooth G-actions on $S^4$ satisfying $\Phi_i \mid K \times S^4 = \psi$. Then the corresponding triples $(S_i, \varphi_i, f_i)$ defined in § 3 are equivalent if $\Phi_i$ are equivalent.

**Fact 6.3.** Let $(S_i, \varphi_i, f_i)$ $(i = 1, 2)$ be triples satisfying the conditions (i) to (vi) in § 4. Then corresponding G-actions $\Phi_i$ $(i = 1, 2)$ constructed by (5.3) are equivalent if $(S_i, \varphi_i, f_i)$ are equivalent.

**Fact 6.4.** Let $\Phi$ be a smooth G-action on $S^4$ satisfying $\Phi \mid K \times S^4 = \psi$ and let $(S, \varphi, f)$ be the triple defined in § 3. Then the G-action $\Phi'$, constructed from $(S, \varphi, f)$ by (5.3), coincides with the given one.

**Fact 6.5.** Let $(S, \varphi, f)$ be a triple satisfying the conditions (i) to (vi) in § 4 and let $\Phi$ be the G-action on $S^4$ constructed from $(S, \varphi, f)$ by (5.3). Then the triple $(S', \varphi', f')$ constructed from $\Phi$ coincides with the given one.

We shall show only the fact 6.3. Let $(S_i, \varphi_i, f_i)$ $(i = 1, 2)$ be equivalent, then there exists a diffeomorphism $\xi : S_1 \to S_2$ such that $\xi J_j = J_j \xi$ $(j = 1, 2)$ and the diagram (6.1) is commutative. Since $\psi \mid K \times S_i : K \times S_i \to S^4$ are smooth, closed and surjective, there exists a K-equivariant homeomorphism $\Psi$ of $S^4$ satisfying $\Psi(\psi(k, X)) = \psi(k, \xi(X))$ for $k \in K$, $X \in S_1$. Now for any $(g, p) \in G \times S^4$, let us choose as in (5.3);$\Phi_1(g, p) = \psi(k_1', \varphi_1(\theta_1, X_1))$, where $p = \psi(k_1, X_1)$ , $gk_1 = k_1'm(\theta_1)u_1$, $u_1 \in H(0{:}b{:}c)$ for $f(X_1) = (0{:}b{:}c)$. Then we have

$$\begin{aligned}\Psi(\Phi_1(g, p)) &= \Psi(\psi(k_1', \varphi_1(\theta_1, X_1))) = \psi(k_1', \xi\varphi_1(\theta_1, X_1)) \\ &= \psi(k_1', \varphi_2(\theta_1, \xi(X_1))) = \Phi_2(g, \Psi(p)).\end{aligned}$$

Thus $\Psi$ is also G-equivariant. Let $S_i(T) = \{X = (u,v,w) \in S_i \mid f_i(X) \neq (0{:}1{:}0), f_i(X) \neq (0{:}0{:}1)\}$. Since $\psi \mid K \times S_i(T)$ are open maps and have smooth local sections, $\Psi$ is also a diffeomorphism on $S^4 - \{B(T^2) \cup B(SU(2))\}$, where $B(T^2) = \{\psi(k, e_3) \mid k \in K\}$ and $B(SU(2)) = \{\psi(k, e_2) \mid k \in K\}$ are two singular orbits of the K-action $\psi$ on $S^4$. On the other hand, open orbits of G-actions $\Phi_i$, $\{\Phi_i(g, e_3) \mid g \in G\}$ and $\{\Phi_i(g, e_2) \mid g \in G\}$, are equivariantly diffeomorphic to $G/H(0{:}0{:}1)$ and $G/H(0{:}1{:}0)$, respectively. Hence the G-equivariant homeomorphisms $\Psi \mid \{\Phi_1(g, e_i) \mid g \in G\} : \{\Phi_1(g, e_i) \mid g \in G\} \to \{\Phi_2(g, e_i) \mid g \in G\}$ $(i = 2, 3)$ become diffeomorphisms naturally. Thus $\Psi$ is a G-equivariant diffeomorphism and hence $\Phi_1$ and $\Phi_2$ are equivalent. This proves Fact 6.3.

By Fact 3.2 and four facts above, we obtain the following main result.

**Theorem.** There is a one-to-one correspondence between the equivalence classes of smooth $Sp(2,\mathbb{R})$-actions on $S^4$ without fixed points and the equivalence classes of triples satisfying the conditions (i) to (vi) in § 4.

## 7. Appendix

Let $(S, \varphi, f)$ be a triple satisfying the conditions (i) to (vi) in § 4. Then we denote

$$F(S, \varphi, f) = \{ X \in S \mid \varphi(\theta, X) = X \text{ for any } \theta \in \mathbb{R}\}.$$

We say that $X_1, X_2 \in F(S, \varphi, f)$ is equivalent if $X_2 = J_1{}^r J_2{}^s(X_1)$ for some $r, s \in \{1, 2\}$ and we denote the set of the equivalence classes by $\{F(S, \varphi, f)\}$. Then we have the next lemma by definition of $(S, \varphi, f)$ .

**Lemma 7.1.** If $\{F(S, \varphi, f)\}$ consists of m elements, then the G-action on $S^4$ constructed from $(S, \varphi, f)$ by (5.3) consists of (2m+1) orbits.

**Example** Let $\Phi_o$ be the standard G-action on $S^4$ introduced in § 2. Then the triple $(S_o, \varphi_o, f_o)$ is as follows.

$$S_o = \{(0,v,w) \in S^2\},\ f_o(0,v,w) = (0:v:w) \text{ and}$$

$$\varphi_o(\theta, (0,v,w)) = (0,v',w'),$$

where $v' = (v \cosh \theta + w \sinh \theta)((v^2 + w^2)\cosh 2\theta + 2vw \sinh 2\theta)^{-1/2}$, $w' = (v \sinh \theta + w \cosh \theta)((v^2 + w^2)\cosh 2\theta + 2vw \sinh 2\theta)^{-1/2}$. Moreover $\{F(S_o, \varphi_o, f_o)\}$ consists of only one element.

Let m be a positive integer and let $S = S^1 = \{(0,v,w)\}$ in $S^2$. Then we can construct a triple $(S, \varphi, f)$ satisfying the conditions (i) to (vi) in § 4 such that $\{F(S, \varphi, f)\}$ consists of (2m-1) elements. Hence we have

**Corollary.** There are infinitely many non-equivalent smooth $Sp(2,\mathbb{R})$-actions on $S^4$ without fixed points.

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［補註1］この報告は同名の論文[9]の要約である。

［補註2］$Sp(2,\mathbf{R})/Z_2 \cong SO_0(2,3)$, $SL(2,\mathbf{R}) \cong SO_0(3,1)$ であることが知られている。また、1.2で定義された $Sp(2,\mathbf{R})$の5次元表現を$\rho$とすると、$\rho$ は同型対応 $Sp(2,\mathbf{R})/Z_2 \cong SO_0(2,3)$を与える。

(都立航空航専)

# Poisson Cohomology of Quadratic Poisson Structures

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## 1. Introduction

As is well-known, Poisson cohomology is of special importance in the theory of Poisson geometry. But unfortunately, the computation is very complicated and only a few methods to compute it are found.

Let $(M, \pi)$ be a poisson manifold, where $M$ is a $C^{\infty}$-manifold and $\pi$ denotes a Poisson structure on $M$. If the rank of $\pi$ is everywhere constant on $M$, $(M, \pi)$ is said to be *regular*. We can find some papers about Poisson cohomology defined on regular Poisson manifolds (see e.g., [4],[8],[10]).

If $(M, \pi)$ is not regular, certain difficulties will arise in computations of Poisson cohomology. A typical example of such manifolds is a *linear Poisson manifold*. It is, by definition, the dual space of a finite dimensional Lie algebra, and its Poisson structure is naturally induced from the Lie algebra structure. There are also some results on the computations of their Poisson cohomology (see e.g., [3],[6],[7]).

In the present article, we shall treat *quadratic* Poisson structures $\pi$ on the plane $R^2$, and compute mainly the first Poisson cohomology. Note that each Poisson manifold $(R^2, \pi)$ is not regular except for the trivial one, i.e. $(R^2, 0)$. In considering this problem, the author was motivated by I.Vaisman's book (see p.67 in [9]).

## 2. Poisson manifolds and Poisson cohomology

Let $(M, \pi)$ be a Poisson manifold. Then $\pi$ is written in local coordinates $(x_1, x_2, ..., x_n)$ as

$$\pi = \frac{1}{2} \sum_{1 \leq i,j \leq n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

with $\pi_{ij} = -\pi_{ji}$, such that $\pi$ satisfies the Jacobi identity

$$\sum_{1 \leq l \leq n} \Big( \pi_{il} \frac{\partial \pi_{jk}}{\partial x_l} + \pi_{jl} \frac{\partial \pi_{ki}}{\partial x_l} + \pi_{kl} \frac{\partial \pi_{ij}}{\partial x_l} \Big) = 0.$$

Since the Poisson bracket is defined by $\{f,g\} =< \pi \mid df \wedge dg >$, the coefficients $\pi_{ij}$ of $\pi$ are obtained by the Poisson bracket: $\pi_{ij} = \{x_i, x_j\}$. If each $\pi_{ij}$ is a homogeneous linear polynomial, then the Poisson structure $\pi$ is said to be *linear*. Similarly if each $\pi_{ij}$ is homogeneous quadratic polynomial, then it is said to be *quadratic*.

From now on, we denote the *Schouten bracket* by $[\, ,\, ]$. One is referred to [9] for the further studies of the Schouten bracket. The space of infinitesimal automorphisms of the Poisson structure $\pi$, which we denote by $Z^1_\pi(M)$, is the set of vector fields $X$ satisfying $[X,\pi] = 0$. We denote the space of Hamiltonian vector fields $X_f$, $(f \in C^\infty(M))$, by $B^1_\pi(M)$. Recall that a Hamiltonian vector field $X_f$ is defined by $X_f(g) = \{f,g\}$ for all $g \in C^\infty(M)$.

Let $\chi^i(M)$ denote the space of $i$-vectors (i.e. skew symmetric contravariant tensor fields of type $(i,0)$), and $L(M) = (\bigoplus_{i=0}^n \chi^i(M), \wedge)$ be the contravariant Grassmann algebra of $M$, where $n$ is the dimension of $M$. In particular, $\chi^0(M) = C^\infty(M)$ and $\chi^1(M)$ is the space of all vector fields on $M$, which we denote by $\chi(M)$. Then $L(M)$ becomes a Lie superalgebra with respect to the Schouten bracket. We define a linear mapping $D : L(M) \longrightarrow L(M)$ by $X \mapsto [\pi, X]$. Since the Poisson structure $\pi$ satisfies $[\pi,\pi] = 0$, the linear mapping $D$ satisfies $D^2 = 0$ and it becomes a coboundary operator. $D$ maps $\chi^i(M)$ into $\chi^{i+1}(M)$. The cohomology with respect to this coboundary operator $D$ is called *Poisson cohomology* and is denoted by $H^*_\pi(M)$. The k-th Poisson cohomology space of $(M,\pi)$ is given by

$$H^k_\pi(M) = \frac{\ker(D : \chi^k(M) \longrightarrow \chi^{k+1}(M))}{\operatorname{im}(D : \chi^{k-1}(M) \longrightarrow \chi^k(M))}.$$

Then the following facts are straightforward:

a) $H^0_\pi(M)$ is the center of the Poisson algebra $(C^\infty(M), \{,\})$. This space is also called the space of *Casimir functions.*

b) $H^1_\pi(M) \cong Z^1_\pi(M)/B^1_\pi(M)$.

## 3. Quadratic Poisson structures on $R^2$

In this section, we classify all quadratic Poisson structures on $R^2$. One should consult the papers [1],[2],[5] for the classification of quadratic Poisson structures under the more general situations. Using the theorem of Z-J.Liu and Ping Xu [5], we know that the only "exact" quadratic Poisson structure on $R^2$ is zero. Hence it is quite easy to classify quadratic Poisson structures on $R^2$.

Let $x$,$y$ be the standard coordinates on $R^2$. Then any quadratic Poisson bracket on $R^2$ is given by $\{x,y\} = ax^2 + bxy + cy^2$, where $a$,$b$ and $c$ are arbitrary constants. The Poisson structure corresponding to the Poisson bracket $\{x,y\} = ax^2 + bxy + cy^2$ is equal to $\pi_\Lambda$ for the triangular r-matrix $\Lambda = K \wedge I$, where $K$ is the matrix in $sl(2,R)$;

$$K = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$$

and $I$ denotes the identity matrix. Namely $\pi_\Lambda$ is given by

$$\pi_\Lambda = (ax^2 + bxy + cy^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

We should first note the following result due to Z-J.Liu and Ping Xu.

**Proposition 3.1** [5]. *Let $\Lambda = K \wedge I$ and $\Lambda' = K' \wedge I$ be two triangular r-matrices. Then two quadratic Poisson structures* $\pi_\Lambda$ and $\pi_{\Lambda'}$ *on $R^2$ are Poisson diffeomorphic if and only if $K' = T^{-1}KT$ for a certain linear isomorphism $T$.*

This proposition indicates that in order to classify all quadratic Poisson structures on $R^2$, we have only to classify $sl(2, R)$ by Jordan form. By this procedure, we obtain the classification of all quadratic Poisson structures on $R^2$.

**Proposition 3.2.** *The following is a complete list of all quadratic Poisson structures $\pi$ on $R^2$ up to a linear isomorphism. (The subscript $\Lambda$ is omitted.)*

(1) $K = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\pi = 0$.

(2) $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\pi = (x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

(3) $K = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$, then $\pi = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

(4) $K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\pi = y^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$.

## 4. Computations of Poisson cohomology

**4.1. The case (1):** $(R^2, \pi = 0)$. The cohomology spaces $H^*_\pi(R^2)$ are easily obtained. In fact, we immediately have $H^*_\pi(R^2) = \chi^*(R^2)$.

For other cases, the following results are useful for computations of Poisson cohomology.

**Proposition 4.1** [9]. *If a Poisson manifold $(M, \pi)$ is a symplectic manifold, that is, if $\pi$ is of full rank, it holds $H^*_\pi(M) \cong H^*_{dR}(M)$, where $H^*_{dR}(M)$ stands for the usual de Rham cohomology.*

**Proposition 4.2** [9]. *If $(M_1, \pi_1)$ and $(M_2, \pi_2)$ are Poisson manifolds and $\phi : M_1 \longrightarrow M_2$ is a Poisson mapping which is a local diffeomorphism, then one has an induced homomorphism $\phi^* : H^k_{\pi_2}(M_2) \longrightarrow H^k_{\pi_1}(M_1)$.*

**4.2. The case (2)**: $(R^2, \pi = (x^2+y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. In this case, it is easy to see that Casimir functions are only constants. Hence we have $H^0_\pi(R^2) \cong R$. We will proceed to compute $H^1_\pi(R^2)$. Since the canonical inclusion mapping $\iota : R^2 - (0) \longrightarrow R^2$ is a Poisson map, by Proposition 4.2, it induces a homomorphism $\iota^* : H^*_\pi(R^2) \longrightarrow H^*_\pi(R^2 - (0))$. Note that $(R^2 - (0), \pi)$ is a symplectic manifold. Hence by Proposition 4.1, we get: $H^1_\pi(R^2 - (0)) \cong H^1_{dR}(R^2 - (0)) \cong R$. Consider a vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Then $[X, \pi] = 0$ and it is easy to check that $[X] \neq 0$ in $H^1_\pi(R^2)$. Moreover $\iota^*[X] \neq 0$ even in $H^1_\pi(R^2 - (0))$, and it generates $H^1_\pi(R^2 - (0))$. Hence the mapping $\iota^* : H^1_\pi(R^2) \longrightarrow H^1_\pi(R^2 - (0))$ is surjective. Put $F = C^\infty(R^2)$, and define a space $\mathcal{F}$ by

$$\mathcal{F} = \{f \in C^\infty(R^2 - (0)) \mid (x^2+y^2)\frac{\partial f}{\partial x},\ (x^2+y^2)\frac{\partial f}{\partial y} \in F\}.$$

Then $\mathcal{F}$ contains $F$ as its subspace. We define a linear mapping $T : \mathcal{F} \longrightarrow H^1_\pi(R^2)$ by $T(f) = [X_f]$. Then it is clear that $T(\mathcal{F})$ =ker $\iota^*$. Let $f = \frac{1}{2}\log(x^2+y^2)$. Then $f$ is an element of $\mathcal{F}$ and $T(f) = [X_f] = [y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]$ is a non-zero element of $H^1_\pi(R^2)$. But $\iota^*[X_f] = 0$ in $H^1_\pi(R^2 - (0))$. Hence ker $\iota^* \neq 0$. The following lemma is quite easy.

**Lemma 4.3.** (a) $H^1_\pi(R^2)/\ker\iota^* \cong H^1_\pi(R^2 - (0)) \cong R$.

(b) $\mathcal{F}/F \cong \ker\iota^*$.

Next we precisely determine the space $\mathcal{F}/F$. A vector field $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ belongs to $Z^1_\pi(R^2)$ if and only if $2(xa+yb) = (x^2+y^2)(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y})$, and a Hamiltonian vector field $X_f$ is given by $X_f = (x^2+y^2)(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y})$. A function $f \in C^\infty(R^2 - (0))$ belongs to $\mathcal{F}$ if and only if $f$ satisfies $(x^2+y^2)\frac{\partial f}{\partial x} = b$, and $(x^2+y^2)\frac{\partial f}{\partial y} = -a$ for some functions $a, b \in F$. By the integrability condition of $f$, it holds

(4.1) $\quad (x^2+y^2)(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}) = 2(xa+yb).$

This equation is just equal to the condition that a vector field $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ coincides with a Hamiltonian vector field $X_f$, where $(x^2+y^2)\frac{\partial f}{\partial x} = b$ and $(x^2+y^2)\frac{\partial f}{\partial y} = -a$.

To prove the following lemma is straightforward but is slightly tedious.

**Lemma 4.4.** *Let $A$ and $B$ be two homogeneous polynomials of degree $n$, $(n \geq 2)$. If $A$ and $B$ satisfy*

(4.1)' $\quad (x^2+y^2)(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}) = 2(xA+yB),$

*then both of them have a factor $x^2+y^2$.*

**Proposition 4.5.** *$\mathcal{F}/F$ is isomorphic to $R$.*

*Proof.* For any $f \in \mathcal{F}$, there exist $a, b \in F$ such that

$$\begin{cases} (x^2+y^2)\frac{\partial f}{\partial x} = b, \\ (x^2+y^2)\frac{\partial f}{\partial y} = -a, \\ (x^2+y^2)(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}) = 2(xa+yb). \end{cases}$$

By Lemma 4.4, there exist $g, h \in F$ such that $a$ and $b$ can be rewritten as follows:

$$\begin{cases} a = a_1 x + b_1 y + (x^2+y^2)\cdot g(x,y), \\ b = -b_1 x + a_1 y + (x^2+y^2)\cdot h(x,y), \\ \frac{\partial g}{\partial x} = -\frac{\partial h}{\partial y}, \end{cases}$$

where $a_1$ and $b_1$ are constants. Put $\alpha = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. Then

$$\begin{aligned} (4.2) \quad \alpha &= a_1 \tfrac{ydx-xdy}{x^2+y^2} - b_1 \tfrac{xdx+ydy}{x^2+y^2} + hdx - gdy \\ &= a_1 \tfrac{ydx-xdy}{x^2+y^2} - d\{\tfrac{b_1}{2}\log(x^2+y^2)\} + d(\textstyle\int hdx). \end{aligned}$$

Since $[\alpha] = 0$ in $H^1_{dR}(R^2-(0))$ and the generator of $H^1_{dR}(R^2-(0))$ is $[\frac{ydx-xdy}{x^2+y^2}]$, we have $a_1 = 0$ in (4.2). Thus from (4.2), it follows that $d\{f + \frac{b_1}{2}\log(x^2+y^2) - \int hdx\} = 0$ and we get $f \equiv -\frac{b_1}{2}\log(x^2+y^2), (\mathrm{mod} F)$. This completes the proof. q.e.d.

Combining Lemma 4.3 and Proposition 4.5, we get the following theorem.

**Theorem 4.6.** $H^1_\pi(R^2) \cong R \oplus R$.

**The case (3):** $(R^2, \pi = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. The space of Casimir functions coincides with $R$. Hence $H^0_\pi(R^2) \cong R$. Put $N = \{(x\text{–axis}) \cup (y\text{–axis})\}$. To compute $H^1_\pi(R^2)$, let us also consider the canonical inclusion $\iota : R^2 - N \longrightarrow R^2$. By Proposition 4.2, we have the induced homomorphism $\iota^* : H^*_\pi(R^2) \longrightarrow H^*_\pi(R^2-N)$. Since $(R^2-N, \pi)$ is a symplectic manifold, it follows $H^1_\pi(R^2-N) \cong H^1_{dR}(R^2-N) = 0$. Thus $\iota^* : H^1_\pi(R^2) \longrightarrow H^1_\pi(R^2-N) = 0$ is clearly surjective. Let $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ be an element of $Z^1_\pi(R^2)$. Then $a$ and $b$ satisfy

$$(4.3) \quad bx + ay = xy(\tfrac{\partial a}{\partial x} + \tfrac{\partial b}{\partial y}).$$

On the other hand, a Hamiltonian vector field $X_f$ is given by

$$(4.4) \quad X_f = xy(-\tfrac{\partial f}{\partial y}\tfrac{\partial}{\partial x} + \tfrac{\partial f}{\partial x}\tfrac{\partial}{\partial y}).$$

Let us define a space $\mathcal{H}$ by

$$\mathcal{H} = \{f \in C^\infty(R^2 - N) \mid xy\frac{\partial f}{\partial x}, xy\frac{\partial f}{\partial y} \in F\}.$$

Then $F$ is a subspace of $\mathcal{H}$. It is clear that the mapping $U : f \in \mathcal{H} \longrightarrow [X_f] \in H^1_\pi(R^2)$ is well-defined. We will prove that this mapping $U$ is surjective. To do this, we need the following lemma. (The proof is omitted.)

**Lemma 4.7.** *If two functions $a, b \in F$ satisfy (4.3), then we have*

$$\begin{cases} \frac{\partial^k a}{\partial x^k}(0,0) = 0, (k \geq 0, k \neq 1), \\ \frac{\partial^k a}{\partial y^k}(0,0) = 0, (k \geq 0), \\ \frac{\partial^k b}{\partial x^k}(0,0) = 0, (k \geq 0), \\ \frac{\partial^k b}{\partial y^k}(0,0) = 0, (k \geq 0, k \neq 1). \end{cases}$$

**Proposition 4.8.** *$U : f \in \mathcal{H} \longrightarrow [X_f] \in H^1_\pi(R^2)$ is surjective.*

*Proof.* Let $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, (a, b \in F)$ be any element of $Z^1_\pi(R^2)$. Then $a$ and $b$ satisfy (4.3). We must find out $f \in \mathcal{H}$ which satisfies

(4.5) $\qquad xy\frac{\partial f}{\partial y} = -a, \quad xy\frac{\partial f}{\partial x} = b.$

The integrability condition of $f$ is just equal to (4.3). By Lemma (4.7), there exist $g, h \in F$ such that

$$\begin{cases} a = a_1 x + xy \cdot g(x,y), \\ b = b_1 y + xy \cdot h(x,y), \\ \frac{\partial g}{\partial x} = -\frac{\partial h}{\partial y}. \end{cases}$$

Then the general solution $f(x,y)$ is given by

(4.6) $\qquad f(x,y) = -a_1 \log|x| + b_1 \log|y| - \int g(x,y)dy,$

where $a_1$ and $b_1$ are arbitrary constants. This function $f(x,y)$ is clearly an element of $\mathcal{H}$, and satisfies (4.5). q.e.d.

**Theorem 4.9.** *If $\pi = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, then $H^1_\pi(R^2) \cong R \bigoplus R$.*

*Proof.* Since $U$ and $\iota^*$ are surjective by Proposition 4.8, $H^1_\pi(R^2)$ is isomorphic to $\mathcal{H}/F$. This space is spanned by $[f]$, $(\bmod F)$ for the functions $f$ defined by (4.6). More precisely, $H^1_\pi(R^2)$ is generated by two vector fields $[x\frac{\partial}{\partial x}]$ and $[y\frac{\partial}{\partial y}]$. q.e.d.

**4.4. The case (4):** $(R^2, \pi = y^2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. It is also clear that $H^0_\pi(R^2) \cong R$. By the same method as the case (3), we have $H^1_\pi(R^2 - (x\text{–axis})) = 0$. Thus $\iota^* : H^1_\pi(R^2) \longrightarrow$

$H^1_\pi(R^2-(x\text{–axis}))=0$ is surjective, where $\iota: R^2-(x\text{–axis}) \longrightarrow R^2$ is the canonical inclusion. Let $X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}$ be an element of $Z^1_\pi(R^2)$. Then $a$ and $b$ satisfy

(4.7) $\quad 2b=y(\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}).$

A Hamiltonian vector field $X_f$ is given by

(4.8) $\quad X_f=y^2(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial}{\partial y}).$

Next we define a space $\mathcal{K}$ by

$$\mathcal{K}=\{f\in C^\infty(R^2-(x\text{–axis}))|y^2\frac{\partial f}{\partial x},y^2\frac{\partial f}{\partial y}\in F\}.$$

Then $\mathcal{K}$ contains $F$ as its subspace. The linear mapping $V: f\in\mathcal{K}\longrightarrow[X_f]\in H^1_\pi(R^2)$ is well-defined and $\ker V=F$.

**Theorem 4.10.** *If $\pi=y^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$, then $H^1_\pi(R^2)\cong\mathcal{K}/F$, and $\mathcal{K}/F$ is infinite dimensional.*

*Proof.* First we prove that $V$ is surjective. Let $X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}$ be any element of $Z^1_\pi(R^2)$. Then we must find a function $f\in\mathcal{K}$ such that

(4.9) $\quad y^2\frac{\partial f}{\partial y}=-a,\quad y^2\frac{\partial f}{\partial x}=b.$

The integrability condition of $f$ is equal to (4.7). Put $f(x,y)=\frac{1}{y^2}\int b dx$. Then the function $f$ is an element of $\mathcal{K}$, and by using the relation (4.7), we know that it satisfies (4.9). Thus the linear mapping $V$ is surjective. Now it is clear that $H^1_\pi(R^2)$ is isomorphic to $\mathcal{K}/F$. Let $c(x,y)$ be any function of $F$ such that $c(x,0)\neq 0$. Then $c(x,y)/y$ is contained in $\mathcal{K}$, but it is not contained in $F$. Hence $\mathcal{K}/F$ is infinite dimensional. q.e.d.

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# Riemannian Submersions from Locally Conformal Kaehler Manifolds

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## 1. Introduction

R. H. Escobals Jr.[3] classified Riemannian submersions from a sphere $S^n$ with connected totally geodesic fibers. By using this classification, R. H. Escobals Jr. [4] and A. Ranjan [10] classified Riemannian submersions from a complex projective space $P_n(\mathbf{C})$ with connected complex totally geodesic fibers.

Let $(M, g, J)$ be a locally conformal Kaehler manifold with the Lee form $\omega$. Here $J$ denotes the complex structure of $M$ and $g$ its Hermitian metric. A locally conformal Kaehler manifold is called a generalized Hopf manifold if the Lee form $\omega$ is parallel. If $M$ is a generalized Hopf manifold, then the leaves of the foliation defined by the Lee form $\omega = 0$ carry a $c$-Sasakian structure $(\phi, \xi, \eta, h)$, where $c = \frac{|\omega|}{2}$. A generalized Hopf manifold is called a $k$-generalized Hopf manifold if every leaf of foliation defined by $\omega = 0$ is of constant $\phi$-sectional curvature $k$. In [6], J. C. Marrero and J. Rocha classified almost Hermitian submersions from a simply connected complete $k$-generalized Hopf manifold with connected totally geodesic fibers. A Hopf manifold $H^n_\lambda$ is a compact homogeneous 1-generalized Hopf manifold which is not simply connected.

We consider a Riemannian submersion with connected complex totally geodesic fibers and an almost Hermitian submersion with connected totally geodesic fibers from a compact homogeneous $k$-generalized Hopf manifold.

## 2. Locally confomal Kaehler manifold

All manifolds considered are assumed to be connected.
Let $M$ be an almost Hermitian manifold with metric $g$, complex structure $J$ and the fundamental 2-form $\Omega$. An almost Hermitian manifold $M$ is called locally conformal Kaehler if its metric is conformally related to a Kaehler metric in some neighborhood of every point of $M$. The locally conformal Kaehler manifold $M$ is characterized by $d\Omega = \omega \wedge \Omega$, where $\omega$ is a globally defined closed 1-form called the Lee form of $M$ [11]. If $\dim M = 2$ we have $d\Omega = 0$, therefore we may suppose $\dim M \geq 4$. Next we define a Lee vector field $B$ by $g(X, B) = \omega(X)$. Let $\nabla$ be the Levi-Civita connection of $g$. On $M$ we have another torsionless linear connection ${}^W\nabla$, defined in [11], called the Weyl connection, which is just the Levi-Civita connection of $g'$ and is given by

$$ {}^W\nabla_X Y = \nabla_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X,Y)B. $$

It is shown in [11] that an almost Hermitian manifold $M$ is a locally conformal Kaehler if and only if there is a closed 1-form $\omega$ on $M$ such that ${}^W\nabla_X J = 0$. This is equivalent to

$$ \nabla_X JY - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B = J\nabla_X Y - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X,Y)JB, $$

where $X$ and $Y$ are vector fields of $M$.

A locally conformal Kaehler manifold $(M, J, g)$ is called a generalized Hopf manifold if the Lee form is parallel, i.e. $\nabla\omega = 0 (\omega \neq 0)$.

Let $(M, J, g)$ be a generalized Hopf manifold. Since $\nabla\omega = 0$, $l = |\omega|$ is constant. We set $c = \frac{l}{2}$, $u = \frac{\omega}{l}$, $v = -u \circ J$, $U = \frac{B}{l}$, $V = JU$. Next, we consider the foliation $\mathcal{K}$ on $M$ given by $\omega = 0$. Let $K$ be a maximally connected integral submanifold of $\mathcal{K}$. Let $i : K \to M$ be the immersion of the leaf $K$ in $M$. We set $\phi = J \circ i_* + (i^*v) \otimes U_{|K}$, $\xi = V_{|K}$, $\eta = i^*v$, $h = i^*g$, then $(K, \phi, \xi, \eta, h)$ is a $c$-Sasakian manifold (cf. [6], [12]).

A generalized Hopf manifold $M$ is called a $k$-generalized Hopf manifold if every leaf $K$ of the foliation $\mathcal{K}$ is of constant $\phi$-sectional curvature $k$, where $(\phi, \xi, \eta, h)$ is the induced $c$-Sasakian structure on $K$ (cf. [6]).

We recall the following result.

**Lemma 1** ([6]). *Let $(M, J, g)$ be a generalized Hopf manifold. Then, $(M, J, g)$ is a $k$-generalized Hopf manifold if and only if*

$$\begin{aligned} R(X, Y, Z, W) &= \frac{(k + 3c^2)}{4}\{g(X', Z')g(Y', W') - g(Y', Z')g(X', W')\} \\ &- \frac{(k - c^2)}{4}\{v(X)v(Z)g(Y', W') - v(Y)v(Z)g(X', W') \\ &+ v(Y)v(W)g(X', Z') - v(X)v(W)g(Y', Z') \\ &+ g(Z', JY')g(JX', W') - g(JX', Z')g(JY', W') \\ &+ 2g(X', JY')g(JZ', W')\}, \end{aligned}$$

*for all vector fields $X, Y, Z$ and $W$ of $M$ and $X', Y'$, $Z'$ and $W'$ are the orthogonal projections of $X, Y, Z$ and $W$, respectively, onto the tangent planes of the leaves of the canonical foliation $\mathcal{K}$.*

In this paper, we consider the case where $M$ is a locally conformal Kaehler manifold which is strongly non-Kaehler in the sense that $d\Omega \neq 0$ (and so $\omega \neq 0$) at every point of $M$.

**Example 1** ([11]). The Hopf manifolds are defined as $H_\lambda^n = (\mathbf{C}^n - \{0\})/\triangle_\lambda, n > 1$, where $\mathbf{C}$ is the complex plane, $\lambda \in \mathbf{C}$, $|\lambda| \neq 0, 1$ and $\triangle_\lambda$ is the cyclic group generated by the transformation $z \mapsto \lambda z, z \in \mathbf{C}^n - \{0\}$. On the manifold $H_\lambda^n$, we consider the Hermitian metric

$$ds^2 = \frac{1}{\sum_{k=1}^n z^k \bar{z}^k} \sum_{j=1}^n dz^j \otimes d\bar{z}^j,$$

where $\{z^1, \ldots, z^n\}$ are complex cartesian coordinates on $\mathbf{C}^n$. The Hopf manifold $H_\lambda^n$ is an example of a compact homogeneous 1-generalized Hopf manifold which is strongly non-Kaehler and not simply connected.

**Example 2** ([12]). Let $(M, \phi, \xi, \eta, g)$ be a Sasakian manifold and $S^1$ the circle with the element of length $\omega$, then $M \times S^1$ is a generalized Hopf manifold of metric $g + \omega \otimes \omega$ and the Lee form $\omega$.

## 3. Riemannian submersion

Let $M$ and $N$ be Riemannian manifolds. By a Riemannian submersion we mean a $C^\infty$ mapping $\pi : M \to N$ such that $\pi$ is of maximal rank and $\pi_*$ preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fiber $\pi^{-1}(x)$ for some $x \in N$. Throughout this paper, $g$ will denote the Riemannian metric on $M$, and $g'$ the Riemannian metric on $N$. Let $\mathcal{V} = Ker\pi_*$ be the vertical subbundle in the tangent bundle $TM$ and $\mathcal{H}$ the complementary orthogonal distribution of $\mathcal{V}$ in $TM$. $\mathcal{H}$ is said to be the horizontal distribution. Let $X$ denote a tangent vector at $y \in M$. Then $X$ decomposes as $\mathcal{V}X + \mathcal{H}X$, where $\mathcal{V}X$ is tangent to the fiber through $y$ and $\mathcal{H}X$ is perpendicular to it. $\nabla$ and $\nabla'$ denote the Levi-Civita connections of $M$ and $N$ respectively. We define tensors $T$ and $A$ associated with the submersion by

$$T_X Y := \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y + \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y,$$

$$A_X Y := \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y + \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y,$$

for any vector fields $X, Y$ of $M$.

A vector field $X$ of $M$ is said to be basic if $X$ is horizontal and $\pi$-related to a vector field $\tilde{X}$ of $N$. Every vector field $\tilde{X}$ of $N$ has a unique horizontal lift $X$ to $M$, and $X$ is basic. We denote it by $X = h.l.(\tilde{X})$.

Let $R$, $R'$ denote the curvature tensors of $M$ and $N$ respectively. Then we have the following result.

**Lemma 2** ([9]). *If $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{H}$ are vector fields of $N$, then*
$R(X,Y,Z,H) = R'(\tilde{X},\tilde{Y},\tilde{Z},\tilde{H}) \circ \pi - 2g(A_XY, A_ZH) + g(A_YZ, A_XH) + g(A_ZX, A_YH).$

Let $(M, J, g)$ and $(N, J', g')$ be almost Hermitian manifolds. A Riemannian submersion $\pi : M \to N$ is called an almost Hermitian submersion if $\pi$ is an almost complex mapping, i.e., $\pi_* \circ J = J' \circ \pi_*$.

We shall give a example of an almost Hermitian submersion $\pi : M \to N$ from a $k$-generalized Hopf manifold $M$.

**Example 3.** Let $S^{2n-1}$ be a unit sphere with standard Sasakian structure $(\phi, \xi, \eta, g)$. Let $S^1$ be a circle with the element of length $\omega$, then $S^{2n-1} \times S^1$ is 1-generalized Hopf manifold of metric $g + \omega \otimes \omega$ and the Lee form $\omega$ (cf.[14]). Let $\alpha : S^{2n-1} \times S^1 \to S^{2n-1}$ be the natural projection and $\beta : S^{2n-1} \to P_{n-1}(\mathbf{C})$ be the Hopf fibration. Then $\pi = \beta \circ \alpha : S^{2n-1} \times S^1 \to P_{n-1}(\mathbf{C})$ is a Riemannian submersion. Let $\tilde{J}$ and $J$ be almost complex structures on $S^{2n-1} \times S^1$ and $P_{n-1}(\mathbf{C})$ respectively. Since $\beta_* \circ \phi = J \circ \beta_*$, we have $\pi_* \circ \tilde{J} = J \circ \pi_*$, therefore $\pi$ is an almost Hermitian submersion. And the fibers are totally geodesic submanifolds of $S^{2n-1} \times S^1$.

We give the results of Marrero and Rocha.

**Lemma 3** ([6]). *Let $\pi : M \to N$ be an almost Hermitian submersion with minimal fibers from a $k$-generalized Hopf manifold $M$. Then $N$ is a Kaehler manifold of constant holomorphic sectionsl curvature $k + 3c^2$.*

We denote by $M(c,k)$ the $c$-Sasakian manifold of constant $\phi$-sectional curvature $k$. We consider the following mappings.
(1). $\tau(c,k,n) : S^{2n-1}(c,k) \times \mathbf{R} \to P_{n-1}(\mathbf{C})$,
where $\tau(c,k,n) = \beta \circ \alpha$, $\alpha : S^{2n-1}(c,k) \times \mathbf{R} \to S^{2n-1}(c,k)$ is the natural projection and $\beta : S^{2n-1}(c,k) \to P_{n-1}(\mathbf{C})$ is the Hopf fibration.
(2). $\pi(c,n,m) : (\mathbf{R} \times \mathbf{C}^{n-1})(c, -3c^2) \times \mathbf{R} \to \mathbf{C}^m$,
(3). $\gamma(c,k,n) : (\mathbf{R} \times \mathbf{D}_{n-1})(c,k) \times \mathbf{R} \to \mathbf{D}_{n-1}$.
Then the mappings $\tau(c,k,n), \pi(c,n,m)$ and $\gamma(c,k,n)$ are almost Hermitian submersions with totally geodesic fibers.

**Theorem 1** ([6]). *Let $M$ be a simply connected complete $k$-generalized Hopf manifold with the Lee form $\omega$. Let $\pi : M \to N$ be an almost Hermitian submersion with connected totally geodesic fibers. Suppose $c = \frac{|\omega|}{2}$, $dimM = 2n$ and $dimN = 2m$.*
*(1). If $k > -3c^2$ then $\pi$ and $\tau(c, k, n)$ are equivalent.*
*(2). If $k = -3c^2$ then $\pi$ and $\pi(c, n, m)$ are equivalent.*
*(3). If $k < -3c^2$ then $\pi$ and $\gamma(c, k, n)$ are equivalent.*

## 4. Results

**Theorem 2** . *Let $M^{2n}$ be a compact homogeneous $k$-generalized Hopf manifold of real dimension $2n$ and $c = \frac{|\omega|}{2}$, then $k + 3c^2 > 0$. If $\pi : M^{2n} \to N^{2m}$ is a Riemannian submersion with connected complex and totally geodesic fibers, then $\pi$ is one of the following classes:*
*(1). $\pi : M^{2n} \to P_{n-1}(\mathbf{C})$ $\quad (n \geq 2)$,*
*(2). $\pi : M^{4n+4} \to P_n(\mathbf{Q})$ $\quad (n \geq 1)$,*
*where $P_{n-1}(\mathbf{C})$ is a complex projective space and $P_n(\mathbf{Q})$ is a quaternionic projective space.*

Since a Hopf manifold $H^n_\lambda$ is a compact homogeneous 1-generalized Hopf manifold, we have the following result.

**Corollary 1** . *If $\pi : H^n_\lambda \to N^{2m}$ is a Riemannian submersion from a Hopf manifold $H^n_\lambda$ with connected complex and totally geodesic fibers, then $\pi$ is one of the following classes:*
*(1). $\pi : H^n_\lambda \to P_{n-1}(\mathbf{C})$ $\quad (n \geq 2)$,*
*(2). $\pi : H^{2n+2}_\lambda \to P_n(\mathbf{Q})$ $\quad (n \geq 1)$.*

In the cace where a Riemannian submersion $\pi$ is an almost Hermitian submersion, we get the following result.

**Theorem 3** . *Let $M^{2n}$ be a compact homogeneous $k$-generalized Hopf manifold. If $\pi : M^{2n} \to N^{2m}$ is an almost Hermitian submersion with connected totally geodesic fibers, then $m = n - 1$ and $N^{2(n-1)}$ is holomorphically isometric to $P_{n-1}(\mathbf{C})$.*

As a corollary we have

**Corollary 2** . *If $\pi : H^n_\lambda \to N^{2m}$ is an almost Hermitian submersion from a Hopf manifold $H^n_\lambda$ with connected totally geodesic fibers, then $m = n - 1$ and $N^{2(n-1)}$ is holomorphically isometric to $P_{n-1}(\mathbf{C})$.*

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# $S^1$-束に横断的な接触構造

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## §1. 序

$\Sigma = \Sigma_g$ を向き付け可能な種数 $g$ の閉曲面、$M(e,g)$ を $\Sigma$ 上の向き付け可能な $S^1$ 束で Euler 数 $e$ のもの、$\pi : M(e,g) \to \Sigma$ を射影とする。$\Sigma$ と $S^1$ に向きを 1 つずつ固定し、それから定まる向きを $M(e,g)$ に与えておく。

向き付けられた 3 次元多様体 $M$ 上の接触構造 $\xi$ が正 (positive) とは $\xi$ を局所的に定義する任意の 1-form $\alpha$ ($\xi = \mathrm{Ker}\ \alpha$) に対して $d\alpha \wedge \alpha > 0$ となるものを言う。

$M(e,g)$ 上の接触構造 $\xi$ で各 fiber に横断的に交わるものを正の横断的接触構造(positive transverse contact structure) と呼ぶ。以後、これを PTC と略記する。

我々の結果は次のものである。

**定理**　$M(e,g)$ 上に PTC $\xi$ が存在するための必要十分条件は
(1) $g=0$ かつ $e \leq -1$, または (2) $g \geq 1$ かつ $e \leq 2g-2$.

この結果は葉層 $S^1$ 束に関する Milnor-Wood の不等式 ([W]) の接触構造版である。$S^1$ 不変な PTC の存在に関する Kamishima-Tuboi の結果 ([K-T]) と不等式の向きが整合しないが、これは Euler 数の規約の相違によるものであり、後述のように我々の規約の方が自然と思われる。

なお、Ghys-Giroux も独立に上記の結果を得ているとのことである。

以下の議論において三松佳彦氏 ([M]) から有益な助言を頂いたことを感謝する。

## §2. 定理の証明

まず solid torus 上の PTC を調べる。

$D = \{u \in \mathbf{C} \mid |u| \leq 1\}$ を単位円板とし、それを同心円に分割する：$D = \cup_{0\leq t\leq 1} C_t$, ここに $C_t = \{|u| = t\}$. 射影を $\pi : D \times S^1 \to D$ とし、$T_t = \pi^{-1}(C_t)$ とおく。$\xi$ を $D \times S^1$ 上の PTC とする。$0 < t \leq 1$ に対し、特性葉層 $\xi|T_t$ は $T_t$ 上の 1 次元葉層を定め、それは holonomy 微分同相写像 $h_t : \pi^{-1}(t) \to \pi^{-1}(t)$, $\pi^{-1}(t) \cong S^1$, で与えられる。$\widetilde{h_t} : \widetilde{\pi^{-1}(t)} \to \widetilde{\pi^{-1}(t)}$, $\widetilde{\pi^{-1}(t)} \cong \mathbf{R}$, を $h_t$ の lift とし、$\rho_t$ を $\widetilde{h_t}$ の translation number とする。$\rho_t$ は 葉層 $\xi|T_t$ のみにより定まる。さらに $\rho_0 = 0$ と定義すれば対応 $t \mapsto \rho_t$ は $0 \leq t \leq 1$ で連続となる。

**命題 A**
(1) 関数 $t \mapsto \rho_t$ は 単調減少。
(2) $h_{t_0}$ が $\pi^{-1}(t_0) \cong S^1$ の rotation と共役となるような $t_0$ において $\rho_t$ は狭義単調減少。特に $t=0$ において狭義単調減少。

**証明の概略** $(r,\theta,z)$ を $D \times S^1$ の上の円柱座標とする。universal cover $D \times \mathbf{R}$ に lift した座標も同じ記号で表す。$\xi$ が標準的な model $\omega = dz + r^2 d\theta$ であれば特性葉層 $\xi|T_t$ は linear となり、$\rho_t$ は常に狭義単調減少となる。一般の $\xi$ に対しそれを定義する 1-form は $C^\infty$ 関数$\beta = \beta(r,\theta,z)$ を用いて $\omega = dz + \beta d\theta$ と書ける。$T_t$ 上の Legendre curve を調べることにより、$t_1 < t_2$ ならば任意の $z \in \mathbf{R} = \widetilde{\pi^{-1}(t_i)} (i=1,2)$ に対し$\widetilde{h_{t_1}}(z) > \widetilde{h_{t_2}}(z)$ がわかり、これから $\rho_{t_1} \geq \rho_{t_2}$ が従う。詳しい議論は Sato-Tuboi([S-T]) を参照のこと。

$\widetilde{h_{t_0}}(z) = z + \rho_{t_0}$ が平行移動とする。このとき $t_0 < t$ ならば $\widetilde{h_{t_0}}(z) = z + \rho_{t_0} > \widetilde{h_t}(z)$ $(z \in \mathbf{R})$ より両者のグラフを描くと、ある $\varepsilon > 0$ に対し $z + \rho_{t_0} - \varepsilon \geq \widetilde{h_t}(z)$ となり、$\rho_{t_0} \geq \varepsilon + \rho_t$ が従う。

ここで後の準備のため、向きづけられた $S^1$ 束の Euler 数について確認しておく。

$\pi : M \to \Sigma$ を向きづけられた閉曲面上の向きづけられた $S^1$ 束とする。この時この $S^1$ 束の Euler 数 $e$ は次のようにして定まる。まず $\Sigma = \Sigma' \cup D$ と分解する。ここに $\Sigma'$ は $\Sigma$ から 2 次元円板の内部 int $D$ を取り除いたものである。$\pi^{-1}(\Sigma') \cong \Sigma' \times S^1$, $\pi^{-1}(D) \cong D \times S^1$ と各々に自明化を与えておく。2次元 torus $\partial\pi^{-1}(D)$ の 1 次元 homology 類として$-[\partial\Sigma'] = [\partial D] + e[S^1]$ のとき、この $S^1$ 束の Euler 数 $= e$ と定める。但し $[\partial\Sigma']$ 及び $[\partial D]$ は各々 $\Sigma'$ 及び $D$ から境界に誘導される向きを与えた homology 類。(§3 参照。)

**命題 B**

$M(e,g)$ 上に PTC があるとする。このとき $e' \leq e$ ならば $M(e',g)$ 上にも PTC が存在する。

**証明の概略**　$\pi : M(e,g) \to \Sigma$ を射影とする。$M(e,g)$ 上の PTC を $\xi$ とする。$\Sigma$ 上に小さい円板 $D$ を取る。このとき命題 A の議論より、特性葉層 $\xi|\partial\pi^{-1}(D)$ の定める homology 類は $[\partial D] - \delta[S^1]$ ($\delta > 0$, 十分小) となる。このとき $\partial\pi^{-1}(\Sigma')$ 上、$[\partial D] - \delta[S^1] = -[\partial\Sigma'] + (-e-\delta)[S^1]$ となる。

$\pi_2 : M(e-1,g) \to \Sigma$ を Euler 数 $= e-1$ の $S^1$ 束とする。$M(e-1,g) = \pi_2^{-1}(D) \cup \pi_2^{-1}(\Sigma')$ と分解しておく。$\pi_2^{-1}(\Sigma') \cong \Sigma' \times S^1 \cong \pi^{-1}(\Sigma')$ と同じ自明化を与えることにより、$\pi_2^{-1}(\Sigma')$ 上に$\xi|\pi^{-1}(\Sigma')$ を与える。$D$ を十分小さく取ることにより、$\xi|\partial\pi_2^{-1}(\Sigma')$ は linear な葉層と仮定して良い。これを $\pi_2^{-1}(D)$ 上に標準的な model を与えて拡張することを考える。

$\mathbf{R}^2 \times S^1$ 上の標準的な接触構造 $\omega = dz + r^2 d\theta$ に対し、$D(\mu) \subset \mathbf{R}^2$ を原点中心、半径 $\mu > 0$ の円板とするとき、$\partial(D(\mu) \times S^1)$ 上の特性葉層の定める homology 類 $= [\partial D(\mu)] - 2\pi_0\mu^2[S^1]$ である。但し $\pi_0$ は円周率。$\partial\pi_2^{-1}(\Sigma')$ 上、$[\partial D] - 2\pi_0\mu^2[S^1] = -[\partial\Sigma'] + (-e+1-2\pi_0\mu^2)[S^1]$. ここで $1 + \delta = 2\pi_0\mu^2$ となるように $\mu$ を選べば、$-[\partial\Sigma'] + (-e+1-2\pi_0\mu^2)[S^1] = -[\partial\Sigma'] + (-e-\delta)[S^1]$ となり $\pi_2^{-1}(D)$ 上に拡張する。

**定理の証明**

$g = 0$ の場合: $e = 0$ の $S^1$ 束 $M(0,0) = S^2 \times S^1$ には PTC は存在しない。これを見るために、PTC が存在したとして $S^2$ 上に 2 点 $p_0, p_1$ を取り、$S^2 = \cup_{0<t<1} C_t \cup \{p_0, p_1\}$, $C_t$ は同心円、と分割する。これに関して命題 A での $\rho_t$ を考える。この時、$\rho_0 = 0 = \rho_1$ とすると対応 $t \mapsto \rho_t$ は連続となる。一方、命題 A より $\rho_t$ はある $t$ において狭義単調減少ゆえ$\rho_1 < 0$ となり矛盾を生ずる。

これと命題 B から $e > 0$ のとき $M(e,0)$ では PTC は存在しないことが分かる。

一方、$e = -1$ のとき $M(-1,0)$ は 3 次元球面 $S^3$ に通常と逆の向きを与えたものであるが、ここには Hopf $S^1$ 束に直交する自然な PTC $\xi_0$ がある (§3 も参照のこと)。よって再び命題 A より $e < 0$ に対し、$M(e,0)$ 上に PTC が存在する。

$g \geq 1$ の場合: まず条件の必要性を示す。$M = M(e,g)$ 上に PTC $\xi$ があるとする。$\Sigma$ の点 $p_0$ を基点とする基本群の生成元を与える $2g$ 個の単純閉曲線$\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ を

取る。

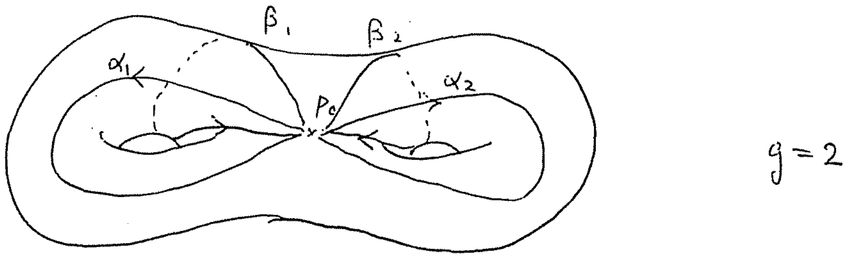


**figure 1**

自明化 $\tau_1 : \widetilde{\pi^{-1}}(\cup_i(\alpha_i \cup \beta_i)) \cong \cup_i(\alpha_i \cup \beta_i) \times \mathbf{R}$ を固定する。

特性葉層 $\xi|\pi^{-1}(\alpha_i)$ の holonomy の $\mathbf{R} \cong \widetilde{\pi^{-1}(p_0)}$ への lift を $\widehat{\alpha}_i$ とする ($\widehat{\beta}_i$ の定義も同様)。交換子積 $\Pi_i[\widehat{\alpha}_i, \widehat{\beta}_i]$ の$\tau_1$ に関する translation number を $\rho$ とする。このとき Wood ([W]) により $1-2g < \rho < 2g-1$ となる。

一方、$M(e,g)$ を $\pi^{-1}(\cup_i(\alpha_i \cup \beta_i))$ で切り開いてできた solid torus を $T$ とし、積構造 $\tau_2 : \widetilde{T} \cong D^2 \times \mathbf{R}$ を固定する。

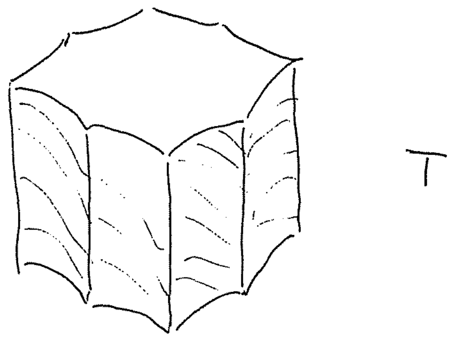


**figure 2**

このとき、$\partial\widetilde{T} = \widetilde{\pi^{-1}}(\cup_i(\alpha_i \cup \beta_i))$ において、

$\tau_1$ に関し水平な模様 $=$ $\tau_2$ に関し $e$ だけの平行移動

なる関係がある。

いま、$\tau_1$ に関する $\xi$ の $\partial\widetilde{T}$ での寄与を$-\delta$ $(\delta > 0)$ とすれば、$\rho + e = -\delta$ となる。このとき $1-2g < \rho = -e-\delta < 2g-1$, よって $1-2g-\delta < e < 2g-1-\delta$ となり $e \leq 2g-2$ が従う。

最後に十分性の概略を示す。命題 B より $e = 2g-2$ なる $M(e,g)$ に対し PTC を構成すれば良い。

$g=1$ の場合、2 次元 Euclid 運動群

$$E(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \theta, a, b \in \mathbf{R} \right\}$$

の上に左不変な PTC を取り、それを $T^3 = \Gamma\backslash E(2)$ に落せばよい。但し

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \middle| m, n \in \mathbf{Z} \right\}.$$

$E(2)$ の Lie algebra の基底

$$E = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

を取れば、$E$ が fiber $S^1$ の方向を与え、さらに Lie bracket の計算から dual basis について $dE^* = 0, dA^* = E^* \wedge B^*, dB^* = -E^* \wedge A^*$ となる。よって左不変な 1-form $\omega = cE^* + aA^* + bB^* (c \neq 0)$ に対し、$\omega \wedge d\omega = (a^2 + b^2)(-E^*) \wedge A^* \wedge B^*$ となり $a = 0 = b$ 以外の $\omega$ は PTC を与える。あるいは $T^3$ の PTC $\omega = dz + (\cos \pi z)dx - (\sin \pi z)dy$ を用いても良い。

$g \geq 2$ では SL$(2;\mathbf{R})$ 上で考えれば良い。Lie algebra $sl(2;\mathbf{R})$ の基底

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

を取る。$E$ が fiber の方向を与える。これらの dual basis について $dH^* = E^* \wedge Q^*, dE^* = H^* \wedge Q^*, dQ^* = -E^* \wedge H^*$ より $\omega = E^* + aH^* + bQ^*$ とおけば$\omega \wedge d\omega = (1 - a^2 - b^2)E^* \wedge H^* \wedge Q^*$ となり、$a, b$ を適当に取れば PTC が得られる。

## §3. 注意 及び 補足

我々の結果に関連する問題を 2 つ挙げる。

問 1.　Seifert fibration $\pi : M^3 \to \Sigma$ に対し PTC の存在条件を決定せよ。

この問題の葉層構造版については最近完全に決定されたようである。従って上記についても決定可能と思われる。

問 2.　向き付けられた 3 次元閉多様体 $M$ の上の非特異ベクトル場 $X$ あるいはそれの定める 1 次元葉層 $\varphi$ に対し PTC の存在条件を決定せよ。

この問題は葉層構造版でも困難である。もしこの問題が解決すれば $X$ あるいは $\varphi$ の "Euler 数" が定義できる。

最後に（蛇足かも知れないが） $S^1$ 束の Euler 数についての補足を述べさせてもらう。まず $S^2$ の単位接円周束 $\pi : M = T_1 S^2 \to S^2$ の Euler 数を調べる。

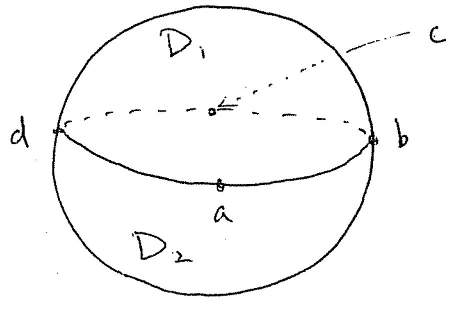


**figure 3**

$S^2 = D_1 \cup D_2$ と 2 つの disk に分割する。$M$ の fiber $S^1$ の向きは $S^2$ の各点の周りに反時計回りとする。このとき $\pi^{-1}(D_1) \to D_1$ の cross section は

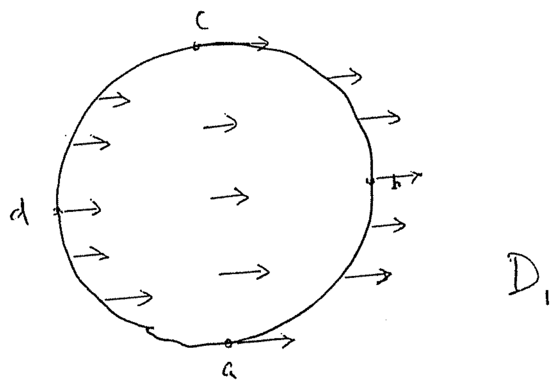


**figure 4**

で与えられる。この cross section の $\partial\pi^{-1}(D_2)$ への制限を$\partial D_2$ の上で描くと

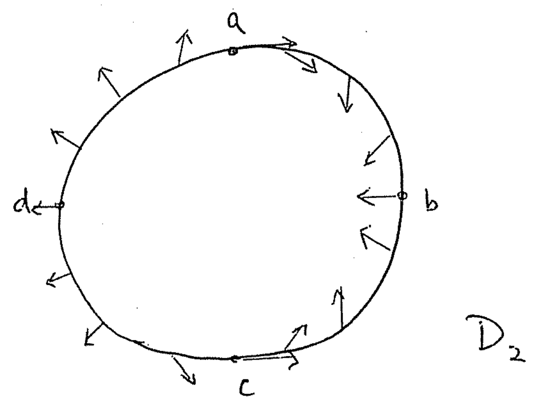


**figure 5**

となる。これを $D_2$ の上の cross section （図は figure 4 と同様）に関してはかると確かに 2 回まわっている。これは $S^2$ の Euler 数 $= 2$ と整合する。

次に $T_1S^2 \cong \mathrm{SO}(3)$ の上の $S^1$ 不変な transverse contact structure について考察する。

$S^2 = \{\, \boldsymbol{x} \in \mathbf{R}^3 \mid \|\boldsymbol{x}\| = 1\}$ に自然な向きを与え、SO(3) の $S^2$ への作用 $(X, \boldsymbol{x}) \mapsto X\boldsymbol{x}$, $X \in \mathrm{SO}(3)$, $\boldsymbol{x} \in S^2$, を考える。このとき

$$p_0 = (0,0,1) \text{ の isotropy 群} = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong T_{p_0}S^2 \text{ の単位円周}$$

と見なす。このとき $\pi : \mathrm{SO}(3) \to S^2$, $\pi(X) = Xp_0$, $X \in \mathrm{SO}(3)$, が左 SO(2) principal bundle となる。

Lie algebra $so(3) = \{\, Y \in M(3) \mid {}^tY + Y = O\}$ の basis

$$E = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

を取ると $(\exp tA)p_0 = (-\sin t, 0, \cos t)$, $(\exp sB)p_0 = (0, -\sin s, \cos s)$ なので$\pi_*A, \pi_*B$

が $T_{p_0}S^2$ の oriented basis となる。

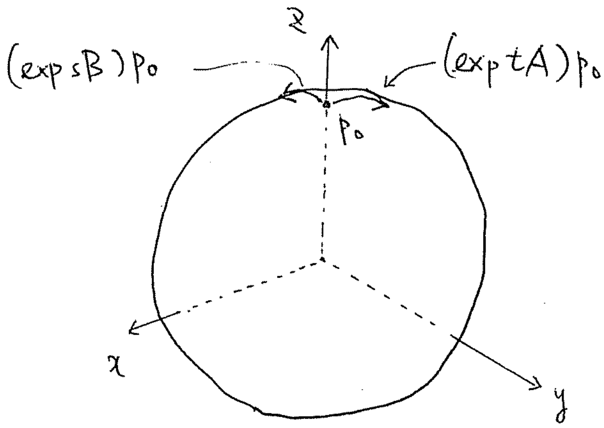


**figure 6**

よって単位元 $I \in \mathrm{SO}(3)$ での接空間 $T_I\mathrm{SO}(3) = so(3)$ にて $A, B, E$ が oriented basis を与える。ここで $E$ が fiber 方向である。SO(3) 上の fiber に横断的な左 SO(2) 不変 1-form $\omega = aA^* + bB^* + E^*$ を取れば、$so(3)$ での関係式 $[A,B] = E, [A,E] = -B, [B,E] = A$ より $dA^* = -B^* \wedge E^*, dB^* = A^* \wedge E^*, dE^* = -A^* \wedge B^*$ となる。これより $d\omega \wedge \omega = (-aB^* \wedge E^* + bA^* \wedge E^* - A^* \wedge B^*) \wedge (aA^* + bB^* + E^*) = -(a^2 + b^2 + 1)A^* \wedge bB^* \wedge E^*$, よって $\omega$ は常に negative である。この double cover を取り、向きを逆にすれば $M(-1, 0)$ の上の PTC $\xi_0$ が得られる。

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# Monge-Ampère equations and surfaces with negative Gaussian curvature

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**§1. Introduction.** In [23], we studied the singularities of solutions of real Monge-Ampère equations of hyperbolic type as follows: Let $z=z(x,y)$ be an unknown function defined for $(x,y)\in\mathbf{R}^2$, then it satisfies

$$F(x,y,z,p,q,r,s,t) = Ar + Bs + Ct + D(rt-s^2) - E = 0 \tag{1.1}$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\partial y$, and $t=\partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of $(x,y,z,p,q)$. As we can see that the singularities of solution $z=z(x,y)$ of (1.1) do not coincide with the singularities of solution surface $\{(x,y,z);\ z=z(x,y)\}$, we would like to study the singularities of solution surfaces of (1.1). Next, as the application, we consider the singularities of surfaces with negative Gaussian curvature. Concerning the second subject, we will give some comments. It is well known that a surface in $\mathbf{R}^3$ with constant negative Gaussian curvature has singular points. (For example, see D. Hilbert [8], E. Holmgren [9].) But we do not know what kinds of singularities may appear. Moreover we would like to extend the solution surface beyond the singularities. Our problems are to answer to the above two questions. As it seems to us that we do not have any result on these problems, we think that, though we assume a little strong conditions, this is one step to construct the global theory on hyperbolic surfaces. We can not treat, by the method developed in this note, the surface whose Gaussian curvature is negative constant, though we will give some comments on this subject in the last section of this note. Here we will give the sketch of our results. The detailed proofs will be given in the forthcoming paper.

**§2. Characteristic method and intermediate integrals.** The aim of this talk is to construct the singularities of the solution surfaces in the case where the equation (1.1) is hyperbolic. For our aim, we have to represent the solutions explicitely. To do so, we apply the characteristic method developed principally by D. Darboux and E. Goursat ([3], [5], [6]). As it seems to us that the method is not familiar today, we will briefly explain it in §2 "from our point of view". But this method depends on a property that the dimension of the space is two. The main idea of Darboux and Goursat is to reduce the solvability of (1.1) to the integration of first order partial didfferential equations, though their theory is local. But, as their method is constructive, it is very useful for our purpose. Let

$$C:\ (x,y,z,p,q) = (x(\alpha),y(\alpha),z(\alpha),p(\alpha),q(\alpha)) \ ,\quad \alpha\in\mathbf{R}^1 \ ,$$

be a smooth curve in $\mathbf{R}^5$, and suppose that it satisfies the following "strip condition"

$$\frac{dz}{d\alpha}(\alpha) = p(\alpha)\frac{dx}{d\alpha}(\alpha) + q(\alpha)\frac{dy}{d\alpha}(\alpha) \quad . \tag{2.1}$$

As a "characteristic strip" means that one can not determine the value of the second derivatives of solution along the strip C, we have the following

**Definition 2.1.** A curve C concerning (x,y,z,p,q) is a "characteristic strip" if it satisfies (2.1) and

$$\det\begin{bmatrix} F_r & F_s & F_t \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} = F_t\,\dot{x}^2 - F_s\,\dot{x}\dot{y} + F_r\,\dot{y}^2 = 0 \tag{2.2}$$

where $F_t=\partial F/\partial t$, $F_s=\partial F/\partial s$, $F_r=\partial F/\partial r$, $\dot{x}=dx/d\alpha$ and $\dot{y}=dy/d\alpha$.

Denote the discriminant of (2.2) by $\Delta$, then

$$\Delta = F_s^2 - 4F_rF_t = B^2 - 4(AC + DE) \quad .$$

If $\Delta < 0$, the equation (1.1) is called to be elliptic. If $\Delta > 0$, the equation (1.1) is hyperbolic. In this note, we will treat the equations of hyperbolic type. More precisely, we assume $\Delta \geq 0$ and also $D\neq 0$. Let $\lambda_1$ and $\lambda_2$ be the solutions of $\lambda^2+B\lambda+(AC+DE)=0$, then the characteristic strip satisfies the following equations:

$$(2.3)\ \left\{\begin{aligned} dz - pdx - qdy &= 0 \\ Ddp + Cdx + \lambda_1 dy &= 0 \\ Ddq + \lambda_2 dx + A\,dy &= 0 \end{aligned}\right. \qquad \text{or} \qquad (2.4)\ \left\{\begin{aligned} dz - pdx - qdy &= 0 \\ Ddp + Cdx + \lambda_2 dy &= 0 \\ Ddq + \lambda_1 dx + A\,dy &= 0 \end{aligned}\right. \quad .$$

**Definition 2.2.** A function V=V(x,y,z,p,q) is called "first integral" of (2.3 ) (or (2.4)) if it is constant on any solution of (2.3) (or of (2.4) respectively).

Let us denote $\omega_0=dz-dx-qdy$, $\omega_1=Ddp+Cdx+\lambda_1dy$ and $\omega_2=Ddq+\lambda_2dx+Ady$. Take an exterior product of $\omega_1$ and $\omega_2$, and substitute into their product the contact relations $\omega_0=0$, $dp=rdx+sdy$ and $dq=sdx+rdy$. Then we get

$$\omega_1\wedge\omega_2 = D\{Ar + Bs + Ct + D(rt - s^2) - E\}\,dx\wedge dy \quad . \tag{2.5}$$

In a space whose dimension is greater than two, the decomposition as above is not possible in general. Assume that the system (2.3) or (2.4) has at least two independent first integrals. We denote them by u and v. Then we can prove that there exists a function $k=k(x,y,z,p,q)\neq 0$ such that

$$(2.6)\qquad du\wedge dv = k\,\omega_1\wedge\omega_2 = k\,D\,\{Ar + Bs + Ct + D(rt - s^2) - E\}\,dx\wedge dy\ .$$

For any function $g$ of two variables whose gradient does not vanish, $g(u,v)=0$ is called an "intermediate integral" of (1.1). Let $C_0$ be an initial strip defined in $\mathbf{R}^5=\{(x,y,z,p,q)\}$. If the strip $C_0$ is not characteristic, we can find an "intermediate integral" $g(u,v)$ which vanishes on $C_0$. Here we put $f(x,y,z,p,q)=g(u,v)$. The Cauchy problem for (1) satisfying the initial condition $C_0$ is to look for a solution $z=z(x,y)$ of (1.1) which contains the strip $C_0$, i.e., two dimensional surface $\{(x,y,z(x,y),\partial z/\partial x(x,y),\partial z/\partial y(x,y))\}$ contains the strip $C_0$. The representation (2.6) means that, as $du\wedge dv=0$ on a surface $g(u,v)=0$, a smooth solution of $f(x,y,z,\partial z/\partial x,\partial z/\partial y)=0$ satisfies the equation (1.1). Therefore we get the following

**Theorem 2.3** ([3], [5]). Assume that the initial strip $C_0$ is not characteristic. Then a solution of the Cauchy problem for (1.1) with the initial condition $C_0$ is locally obtained as a solution of $f(x,y,z,\partial z/\partial x,\partial z/\partial y)=0$ satisfying the same initial condition $C_0$ as for (1.1).

**§3. How to construct examples.** When a partial differential equation of first order

$$(3.1)\qquad f(x,y,z,p,q)=0$$

is given, we will construct Monge-Ampère equation whose intermediate integral coincides with $f(x,y,z,p,q)=0$. Assume $(\operatorname{grad} f)\neq 0$. Then we can locally find a function $g=g(x,y,z,p,q)$ satisfying

$$(3.2)\qquad \operatorname{rank}\begin{pmatrix}\operatorname{grad} f\\ \operatorname{grad} g\end{pmatrix} = 2\ .$$

Here we take a product $df\wedge dg$, substitute there the contact relations $\omega_0=0$, $dp=rdx+sdy$ and $dq=sdx+rdy$, and rewrite it as

$$(3.3)\qquad df\wedge dg = F(x,y,z,p,q,r,s,t)\ \ dx\wedge dy\ .$$

Then we see that the equation $F=0$ has two independent first integrals $f$ and $g$. But we can not globally construct the function $g=g(x,y,z,p,q)$ satisfying the property (3.2), though we can do so for equations of certain types.

**Example 1.** Assume that f is of Hamilton-Jacobi type, i.e., $f=p+h(x,y,z,q)$. Then we choose the function $g$ as $g=q+k(x,y,z)$. Then the equation $F=0$ is obtained by

$$(3.4)\qquad df\wedge dg = \{(rt-s^2)+Ar+Bs+Ct-E\}dx\wedge dy \ = \ F(x,y,z,p,q,r,s,t)\ dx\wedge dy$$

where A, B, C and E are certain functions of $(x,y,z,p,q)$.

**Example 2.** Assume that $f$ is quasi-linear, i.e., $f=ap+bq+c$ where a, b and c are real smooth functions of (x,y,z) and $(a,b)\neq(0,0)$. Then we choose $g=-bp+aq+c'$ where c' is an arbitrary function of (x,y,z). Then the equation $F=0$ is obtained by

$$(3.5) \qquad df\wedge dg = \{(a^2+b^2)(rt-s^2)+Ar+Bs+Ct-E\}\, dx\wedge dy \;=\; F(x,y,z,p,q,r,s,t)\, dx\wedge dy$$

where A, B, C and E are functions of (x,y,z,p,q).

**§4. Introduction to surfaces with negative Gaussian curvature.** Let $\kappa$ be Gaussian curvature of the surface $z=z(x,y)$, then $z=z(x,y)$ satisfies the following equation:

$$(4.1) \qquad rt - s^2 = \kappa\,(1 + p^2 + q^2)^2 .$$

We use the same notations as in §1. Here we are interested in the hyperbolic type, i.e., we assume $\kappa<0$. Lt us remember the classical theorem due to D. Hilbert as follows:

**Theorem** ([8]) A surface S in $\mathbb{R}^3$ with constant negative Gaussian curvature has singular points.

Therefore, when we may extend the classical solution of (4.1), the singularities may appear in general. But, if the Gaussian curvature is not strictly negative, there exists a surface in the large whose Gaussian curvature is negative. See §6 in this note. Here it would be better to make clear the meaning of "singularity".

**Definition 4.1.** A point $(x^0,y^0,z^0)$ is called to be "singularity" of a solution $z=z(x,y)$ if and only if $z^0=z(x^0,y^0)$ and $z(x,y)\notin C^2$ in a neighbourhood of $(x^0,y^0)$.

**Definition 4.2.** Let S be a surface in $\mathbb{R}^3$. S is regular at a point $(x^0,y^0,z^0)$ if we can choose parameters $(\alpha,\beta)\in\mathbb{R}^2$ as follows: $x=x(\alpha,\beta)$, $y=y(\alpha,\beta)$ and $z=z(\alpha,\beta)$ satisfy the two conditions

(i) $(x(\alpha^0,\beta^0),y(\alpha^0,\beta^0),z(\alpha^0,\beta^0))=(x^0,y^0,z^0)$, and $x=x(\alpha,\beta)$, $y=y(\alpha,\beta)$ and $z=z(\alpha,\beta)$ are in $C^2$ in a neighbourhood of $(\alpha^0,\beta^0)$,

(ii) $$\operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial\alpha}, \dfrac{\partial y}{\partial\alpha}, \dfrac{\partial z}{\partial\alpha} \\ \dfrac{\partial x}{\partial\beta}, \dfrac{\partial y}{\partial\beta}, \dfrac{\partial z}{\partial\beta} \end{pmatrix} = 2 .$$

**Definition 4.3.** A point $(x^0,y^0,z^0)$ is called to be "singularity" of a solution surface S if and only if S is not regular at the point $(x^0,y^0,z^0)$.

First we show that the singularities of solutions do not generally coincide with those of solution surfaces. Concerning the solution surface $S=\{(x,y,z);\, z=z(x,y)\}$, the problems which we are interestec in are as follows: I) What kinds of singularities may appear ?, and II) Can we extend the solution surfac beyond the singularities ? For the problem II), we have two directions. After the appearance of singularities, the solution $z=z(x,y)$ takes in general sevaral values. One way is to introduce a physical point of view. Then a solution must be single-valued. For this aim, we cut off some parts of solution so that it could become a single-valued weak or generalized solution satisfying the entropy condition for equations of conservation law or the semi-concavity condition for Hamilton-Jacobi equations. By this procedure, the singularities may appear in the solution. See [21], [22], [10], [11], [13], [18] and [19]. The another way is to consider the above problem from geometric point of view. Then we must accept multi-valued solutions. As Monge-Ampère equations appear often in geometric problems, we should take here the second approach. This means that, without cutting off some part of solution surface, we should accept the whole surface of solution and consider the singularities of surface in the meaning of Definition 4.2. This is the subject of §5.

## §5. Solution surfaces of Monge-Ampère equation (1.1).

**Theorem 5.1.** Assume the following two conditions: 1) (2.3) or (2.4) has two independent first integrals, and 2) The intermediate integral is quasi-linear with respect to $(p,q)$. Then the smooth solution surface of (1.1) exists in the large, though the solution $z=z(x,y)$ has singularities.

**(Idea of Proof)** By the assumption, we can write the intermediate integral $f = f(x,y,z,p,q)$ as

$$(5.1) \qquad \begin{cases} f(x,y,z,p,q) = ap + bq - c = 0 \\ C_0\colon (x,y,z,p,q) = (x_0(\alpha),y_0(\alpha),z_0(\alpha),p_0(\alpha),q_0(\alpha)) \end{cases}$$

The characteristic equations for (5.1) is written by

$$(5.2) \quad \frac{dx}{d\beta} = a\ , \quad \frac{dy}{d\beta} = b\ , \quad \frac{dz}{d\beta} = c \quad ; \quad x(0) = x_0(\alpha)\ , \quad y(0) = y_0(\alpha)\ , \quad z(0) = z_0(\alpha)\ .$$

We denote the solutions of (5.2) by $x=x(\alpha,\beta)$, $y=y(\alpha,\beta)$ and $z=z(\alpha,\beta)$, then we can prove

$$\operatorname{rank} \begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\[2ex] \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = 2 \quad \text{for any } (\alpha,\beta) \in \mathbf{R}^2\ .$$

Therefore, though the solution $z=z(x,y)$ has singularities at the points where the Jacobian $D(x,y)/D(\alpha,\beta)=0$, the solution surface is regular even at these points.

**q.e.d.**

**Theorem 5.2.** Assume the following two conditions: 1) (2.3) or (2.4) has two independent first integrals, and 2) The intermediate integral is of Hamilton-Jacobi type. Then $z=z(x,y)$ is singular at a point $(x^0,y^0)$ if and only if the solution surface $\{(x,y,z); z=z(x,y)\}$ is not regular at a point $(x^0,y^0,z^0)$ where $z^0=z(x^0,y^0)$. Moreover we can uniquely extend the solution surface beyond the singularities in the space of $C^1$-functions which are of class $C^2$ except on piecewise smooth curves.

**Remark.** Let us explain the meaning of "Hamilton-Jacobi type". In [22], we have discussed the differences between Hamilton-Jacobi equations and equations of conservation law, under the assumption that $f(x,y,z,p,q)$ is smooth. Our conclusion is that the most characteristic property of Hamilton-Jacobi equations is the global solvability of the Cauchy problem for (5.4) given a little later. On the other hand, if $f=0$ is quasi-linear, the solution $p(\alpha,\beta)$ and $q(\alpha,\beta)$ tend to infinity when the Jacobian $D(x,y)/D(\alpha,\beta)$ vanishes. Therefore, in the above theorem, "Hamilton-Jacobi type" means the global solvability of the Cauchy problem for (5.4). Recently S. Izumiya ([11]) gave the geometric characterization of Hamilton-Jacobi equations and quasi-linear partial differential equations of first order.

**(Idea of Proof)** By the assumption, we can write the intermediate integral $f = f(x,y,z,p,q)$ as

$$(5.3)\qquad \begin{cases} f(x,y,z,\partial z/\partial x,\partial z/\partial y) = \partial z/\partial x + h(x,y,z,\partial z/\partial y) = 0 \\ z(x(\alpha),y(\alpha)) = z(\alpha)\,,\ (\partial z/\partial x)(x(\alpha),y(\alpha)) = p(\alpha),\ (\partial z/\partial y)(x(\alpha),y(\alpha)) = q(\alpha) \end{cases}$$

The characteristic differential equations for (5.3) are written as follows:

$$(5.4)\qquad \begin{cases} \dfrac{dx}{d\beta} = \dfrac{\partial f}{\partial p}(x,y,z,p,q), \qquad \dfrac{dy}{d\beta} = \dfrac{\partial f}{\partial q}(x,y,z,p,q) \\ \dfrac{dz}{d\beta} = p\dfrac{\partial f}{\partial p}(x,y,z,p,q) + q\dfrac{\partial f}{\partial q}(x,y,z,p,q) \\ \dfrac{dp}{d\beta} = -\dfrac{\partial f}{\partial x}(x,y,z,p,q) - p\dfrac{\partial f}{\partial z}(x,y,z,p,q), \quad \dfrac{dq}{d\beta} = -\dfrac{\partial f}{\partial y}(x,y,z,p,q) - q\dfrac{\partial f}{\partial z}(x,y,z,p,q) \end{cases}$$

$$(5.5)\qquad x(0)=x(\alpha),\ y(0)=y(\alpha),\ z(0)=z(\alpha),\ p(0)=p(\alpha),\ q(0)=q(\alpha)\,, \qquad \alpha\in\mathbb{R}^1\,.$$

We denote the solutions of (5.4)-(5.5) by $x=x(\alpha,\beta)$, $y=y(\alpha,\beta)$, $z=z(\alpha,\beta)$, $p=p(\alpha,\beta)$ and $q=q(\alpha,\beta)$. As $\omega_0=0$ on the solution surface, we have

$$(5.6)\qquad \operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial y}{\partial\alpha} & \dfrac{\partial z}{\partial\alpha} \\ \dfrac{\partial x}{\partial\beta} & \dfrac{\partial y}{\partial\beta} & \dfrac{\partial z}{\partial\beta} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial y}{\partial\alpha} \\ \dfrac{\partial x}{\partial\beta} & \dfrac{\partial y}{\partial\beta} \end{pmatrix} \qquad \text{for any } (\alpha,\beta)\in\mathbb{R}^2\,.$$

The solution $z=z(x,y)$ has singularities at the points where the Jacobian $D(x,y)/D(\alpha,\beta)=0$. The equation (5.6) means that the solution surface is not also regular at the points where the Jacobian vanishes. From these ideas, we can arrive at the above conclusion. **q.e.d.**

**§6. Example.** Assume the Gaussian curvature $\kappa = -1/(1 + p^2 + q^2)^2$. Then the equation (4.1) is written as

$$rt - s^2 = -1 \quad . \tag{6.1}$$

The first integrals of (6.1) is obtained by $\{x+q, y-p\}$. Let the initial strip $C_0$ be

$$C_0: \ (x,y,z,p,q) = (0, \alpha, 1/2\, \alpha^2, 0, \alpha), \qquad \alpha \in \mathbf{R}^1 \quad . \tag{6.2}$$

Then the intermediate integral $g=g(x,y,z,p,q)$ corresponding to this Cauchy problem is given by $g=p+q+x-y$. Therefore the solution of (6.1) satisfying the initial condition $C_0$ is written by $z = - 1/2\, x^2 + 1/2\, y^2$. This means that there exists a smooth surface in the large whose Gaussian curvature is equal to $-1/(1 + p^2 + q^2)^2$.

**§7. Surfaces with strictly negative Gaussian curvature.** As the generalization of Hilbert's theorem ([8]), N. V. Efimov proved the following

**Theorem** ([4]) No surface can be immersed in Euclidean 3-space so as to be complete in the induced Riemannian metric, with strictly negative Gaussian curvature.

We write $\lambda_1 = (1 + p^2 + q^2)\,(-\kappa)^{1/2}$, $\lambda_2 = - \lambda_1$, $\omega_1=dp+\lambda_1 dy$ and $\omega_2=dq+\lambda_2 dx$. Then we have on $\{\omega_0=dz-dx-qdy=0, dp=rdx+sdy\ , \ dq=sdx+rdy\}$

$$\omega_1 \wedge \omega_2 = \{ (rt - s^2) - \kappa\, (1 + p^2 + q^2)^2 \}\, dx \wedge dy\ .$$

Therefore the equation (4.1) is obtained by the product of $\omega_1$ and $\omega_2$. Here we would be better to rewrite the Definition 2.2 as follows:

**Definition 7.1.** A function $V=V(x,y,z,p,q)$ is called "first integral" of $\{\omega_0, \omega_1, \omega_2\}$ if $dV\equiv 0 \mod \{\omega_0, \omega_1, \omega_2\}$.

Then we get

**Theorem 7.2.** Assume that the Gaussian curvature $\kappa$ is strictly negative, then the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ does not have two independent first integrals.

Therefore we can not apply our preceding method to solve (4.1). Then our problem is how we can get a family of characteristic strip. We can obtain it by solving a system of first order partial differential equations:

$$
(7.1) \qquad \left\{
\begin{aligned}
\frac{\partial p}{\partial \alpha} + \lambda_1 \frac{\partial y}{\partial \alpha} &= 0 \\
\frac{\partial q}{\partial \alpha} - \lambda_1 \frac{\partial x}{\partial \alpha} &= 0 \\
\frac{\partial p}{\partial \beta} - \lambda_1 \frac{\partial y}{\partial \beta} &= 0 \\
\frac{\partial q}{\partial \beta} + \lambda_1 \frac{\partial x}{\partial \beta} &= 0
\end{aligned}
\right.
$$

The local solvability of (7.1) is already proved by H. Lewy([13]) and J. Hadamard ([7]). But, to develop the global theory, we would have to consider the global behaviour of the solutions of (7.1).

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# LIE SPHERE GEOMETRY における曲線と曲面

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## 1. Lie sphere geometry.

### 1.1 Lie spheres and Lie transformations.

Let $T_1S^n$ be the unit tangent bundle of the unit sphere $S^n$, that is, $T_1S^n = \{(u,v) \in S^n \times S^n \ ; \ u \cdot v = 0\}$, where $\cdot$ denotes the inner product of the Euclidean space $\mathbf{E}^{n+1}$. Let $\iota\colon S^{n-1} \to S^n$ be a hypersphere with a unit normal vector field $\xi\colon S^{n-1} \to S^n$ along $\iota$, which gives an orientation of $\iota$. We consider a map $(\iota,\xi)\colon S^{n-1} \to T_1S^n$, which we call also an *oriented hypersphere*. When $\iota$ shrinks to a point, we let $\xi$ be the inclusion of the fiber of $T_1S^n$ over the point $\iota$ into $T_1S^n$. We call $(\iota,\xi)$ a *point sphere*. We use the term *Lie sphere* to denote an oriented hypersphere or a point sphere.

**Definition.** A diffeomorphism $T_1S^n \to T_1S^n$ is called a *Lie transformation* if it carries Lie spheres to Lie spheres.

For example a *Möbius transformation* and a *parallel transformation* are Lie transformations; the former takes point spheres to point spheres and the latter takes $(u,v) \in T_1S^n$ to $(\cos tu + \sin tv,\ -\sin tu + \cos tv) \in T_1S^n$, where $t \in [0,\pi)$. It is known that Lie transformations are generated by Möbius transformations and parallel transformations. (See [CC, Theorem 3.1].)

Let $\mathbf{R}^{n+3}_2$ be an (n+3)-dimensional real vector space with an indefinite scalar product $\langle\ ,\ \rangle$ with signature $(n+1,2)$. In this paper we let $\langle x,y\rangle = {}^t xSy$ for $x,y \in \mathbf{R}^{n+3}_2$, where we take

$$S = (S_{ij}) = \begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_{n-1} & 0 \\ -I_2 & 0 & 0 \end{pmatrix}.$$

We denote by $\mathbf{P}^{n+2}$ the associated projective space, and by $\mathbf{Q}^{n+1}$ the quadric in $\mathbf{P}^{n+2}$ defined by $\langle x,x\rangle = 0$. Then we can identify a Lie sphere in $T_1S^n$ with a point of $\mathbf{Q}^{n+1}$. Let $\Lambda^{2n-1}$ be the space of all projective lines which lie on $\mathbf{Q}^{n+1}$. Then it is known that

$$T_1S^n \cong \Lambda^{2n-1}.$$

A Lie transformation can be regarded as a diffeomorphism $\mathbf{Q}^{n+1} \to \mathbf{Q}^{n+1}$ preserving lines on $\mathbf{Q}^{n+1}$, that is the restriction of a projective transformation $\mathbf{P}^{n+2} \to \mathbf{P}^{n+2}$ preserving $\mathbf{Q}^{n+1}$. Thus,

$$\text{the group of all Lie transformations} \cong PO(n+1,2) = O(n+1,2)/\{\pm 1\},$$

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where $O(n+1,2) = \{P \in GL(n+3;\mathbf{R}) \ ; \ {}^tPSP = S\}$. The group $PO(n+1,2)$ acts on $\Lambda^{2n-1}$ transitively. Then $T_1S^n$ is equal to a homogeneous space $PO(n+1,2)/H$ for the isotropy subgroup $H$ of $PO(n+1,2)$ at a point.

*Remark.* Let $\Omega = (\omega^i_j)$ be the Maurer-Cartan form on $O(n+1,2)$. Then the manifold $\Lambda^{2n-1}$ has a contact structure with a contact form $\omega^1_{n+3}$, that is, $\omega^1_{n+3} \wedge (d\omega^1_{n+3})^{n-1}$ does not vanish on $\Lambda^{2n-1}$. The condition $\omega^1_{n+3} = 0$ defines a $(2n-2)$-dimensional distribution $D$ on $\Lambda^{2n-1}$ which has integral submanifolds of dimension $n-1$.

### 1.2 Legendre maps and Dupin hypersurfaces.

**Definition.** An immersed hypersurface $f\colon M^{n-1} \to S^n$ with a unit normal field $\xi\colon M^{n-1} \to S^n$ naturally induces a map $\lambda = (f,\xi)\colon M^{n-1} \to T_1S^n$. This map $\lambda$ is called the *Legendre map* induced by $f$ with $\xi$.

Let $\lambda\colon M^{n-1} \to T_1S^n$ and $\tilde{\lambda}\colon \tilde{M}^{n-1} \to T_1S^n$ be embedded Legendre maps. We say that $\lambda$ and $\tilde{\lambda}$ are *Lie equivalent* if $\lambda$ and $\tilde{\lambda}$ are $PO(n+1,2)$-congruent, that is, there exists an element $P \in PO(n+1,2)$ such that $P\lambda(M)$ agree with $\tilde{\lambda}(\tilde{M})$.

A Legendre map is a Legendre submanifold of the contact manifold $T_1S^n$, that is, an immersed $(n-1)$-dimensional integral submanifold of the contact distribution $D$. Conversely a Legendre submanifold $\lambda = (f,\xi)\colon M^{n-1} \to T_1S^n$ naturally induces a smooth map $f\colon M^{n-1} \to S^n$, which may have singularities; however, U. Pinkall shows that the possible singularities of $f$ can always removed by a parallel transformation and that locally a Legendre submanifolds is transformed to be a Legendre map. (See [P, Theorem 1].)

Let $Y$ and $Z$ are smooth maps from $M^{n-1}$ into $\mathbf{Q}^{n+1}$. By $\mathrm{Line}\{Y(p), Z(p)\}$ we denote the line generated by the points $[Y(p)]$ and $[Z(p)]$ in $\mathbf{Q}^{n+1}$ for $p \in M^{n-1}$.

**Definition.** Let $\lambda = \mathrm{Line}\{Y, Z\}\colon M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold. We call the sphere

$$[K(p)] = [rY(p) + sZ(p)]$$

a *curvature sphere* of $\lambda$ at $p \in M^{n-1}$, if there exist a non-zero vector $X$ in $T_pM^{n-1}$ and $r, s \in \mathbf{R}$ with $(r,s) \neq (0,0)$ such that

$$r\ dY(X) + s\ dZ(X) \in \mathrm{Span}\{Y(p), Z(p)\}.$$

A curvature sphere is invariant under Lie transformations. The vector $X$ above is called a *principal vector* corresponding to $[K]$. At each point $p \in M^{n-1}$, there are at most $n-1$ distinct curvature spheres $[K_1], \ldots, [K_g]$. The principal vectors corresponding to the curvature sphere $[K_i]$ form a subspace $T_i$ of the tangent space $T_pM^{n-1}$, and $T_pM^{n-1} = T_1 \oplus \cdots \oplus T_g$. If the dimension of a $T_i$ (which we call the *multiplicity* of $[K_i]$) is constant on an open subset $U$ of $M^{n-1}$, then the distribution $T_i$ is integrable on $U$. A connected submanifold $\mathcal{S}$ of $M^{n-1}$ is called a *curvature surface* if the tangent space $T_p\mathcal{S}$ is equal to a $T_i$ at each point $p \in \mathcal{S}$.

**Definition.** A Legendre submanifold $\lambda\colon M^{n-1} \to \Lambda^{2n-1}$ is called a *Dupin submanifold* if along each curvature surface the corresponding curvature sphere is constant.

**2. Frenet frames.**

Let $N$ be a smooth $n$-dimensional manifold and $G$ a transitive transformation group on $N$. Let $\lambda\colon M \to N$ and $\tilde{\lambda}\colon \tilde{M} \to N$ be embedded submanifolds of dimension $m$.

(1) We say that $\lambda$ and $\tilde{\lambda}$ have *contact of at least order* $k$ at $p \in M$ and $\tilde{p} \in \tilde{M}$ if $\lambda$ and $\tilde{\lambda}$ agree up to the differentials of order $k$ at $p$ and $\tilde{p}$.

(2) We say that $\lambda$ and $\tilde{\lambda}$ have *$G$-contact of at least order* $k$ at $p$ and $\tilde{p}$ if there exists an element $P \in G$ such that $\lambda$ and $P \circ \tilde{\lambda}$ have contact of at least order $k$ at $p$ and $\tilde{p}$.

(3) We say that $\lambda$ and $\tilde{\lambda}$ are *$G$-congruent* if there exist an element $P \in G$ such that $P\lambda(M) = \tilde{\lambda}(\tilde{M})$.

We fix an origin $o \in N$. We denote the isotropy subgroup of $G$ at $o$ by $H$, then the map $\pi\colon G \to N$ defined by $\pi(P) = P(o)$ induces a diffeomorphism $G/H \cong N$. Let $\lambda\colon M \to G/H \cong N$ be a connected, smoothly embedded $n$-dimensional submanifold of a homogeneous space $G/H$. We state the definition of Frenet frames of $\lambda$ and its construction according with G. R. Jensen [J].

Firstly we construct the set of zeroth order frames. A *zeroth order frame* at $p \in M$ is an element $P \in G$ such that $\pi(P) = \lambda(p)$. Let $L_0$ denote the set of all zeroth order frames on $M$. A *zeroth order frame field* $u$ along $\lambda$ is a smooth cross section of $L_0 \to M$.

*Remark.* Choosing a basis $e_1, \dots, e_m$ of the tangent space $T_oN$, we have a natural bundle map $h\colon G \to L(N)$ defined by $h(P) = P_*(e_1, \dots, e_m)$, where $L(N) \to N$ is the principal $GL(m, \mathbf{R})$-bundle of linear frames on $N$. We identify $P \in G$ with $h(P) \in L(N)$, so we call $P$ a frame.

Secondly we construct first order frames. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of $G$ and $H$ respectively. If $H$ is compact, as a vector subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$ we can take a $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{m}$. With respect to a chosen basis of $\mathfrak{m}$, we consider the linear isotropy representation $\rho\colon H \to GL(m, \mathbf{R})$ given by the adjoint action of $H$ on $\mathfrak{m}$. There is a naturally defined smooth map $\lambda_0$ from $L_0$ to the Grassmann manifold $G_{m,n}$ given by $\lambda_0(P) = P_*^{-1}\lambda_* M_p$ where $\lambda(p) = \pi(P)$. We choose a local cross section $W_1$ of the action $(H, \rho)$ on $G_{m,n}$. We say that $\lambda$ has the type of $W_1$ if there exists a zeroth order frame field $u$ along $\lambda$ such that $\lambda_0(u) \subset W_1$. If $\lambda$ has the type of $W_1$, we let $L_1 = \lambda_0^{-1}(W_1)$ and call $L_1$ the set of *first order frames* on $\lambda$ (with respect to $W_1$). We define a *first order frame field* along $\lambda$ to be a smooth cross section $L_1 \to M$.

The smooth map $\lambda_0 \circ u\colon M \to W_1$ does not depend on the choice of first order frame field $u$ along $\lambda$. Choose a coordinate system $x^1, \dots, x^{\mu_1}$ on $W_1$, where $\mu_1 = \dim W_1$. We call the functions $k^i = x^i \circ \lambda_0 \circ u$ $(i = 1, \dots, \mu_1)$ the *first order invariants* of $\lambda$.

The following proposition shows that the set of first order frames $L_1$ gives first order contact under the action of $G$: (See [J, I.6 Theorem 1].)

**Proposition 2.1.** *let $\lambda\colon M \to G/H$ and $\tilde{\lambda}\colon \tilde{M} \to G/H$ be smoothly embedded $n$-dimensional submanifold on which first order frames can be constructed. Then $\lambda$ and $\tilde{\lambda}$ have $G$-contact of at least order 1 at $p \in M$ and $\tilde{p} \in \tilde{M}$ if and only if they* *are both the type of a local cross section $W_1$ of $\rho$, and they have the same first order invariants at $p$ and $\tilde{p}$.*

Furthermore we iterate this construction of frames and reduce the set of frames; $L_0 \supset L_1 \supset L_2 \supset \cdots$. Thus we construct a set of $k$-th order frames $L_k$ which gives $k$-th order contact under the action of $G$. The sequence $\dim L_0 \geq \dim L_1 \geq \dim L_2 \geq \cdots$ eventually stabilizes. Thus there is the smallest integer $q \geq 1$ such that $\dim L_k = \dim L_q$ for all $k \geq q$. Then the frames of order $q$ are called the *Frenet frames* of $\lambda$.

We also have the following congruence and existence theorem: (See [J, I.11 Theorem 3].)

**Theorem 2.2.** *Let $\lambda\colon M \to G/H$ and $\tilde{\lambda}\colon \tilde{M} \to G/H$ be smoothly embedded $n$-dimensional submanifolds. Then $\lambda$ and $\tilde{\lambda}$ are $G$-congruent if and only if*

(1) *Their Frenet frames are of the same order $q$,*
(2) *they are both the type of a local cross section $W_q$,*
(3) *there exist a one-to-one correspondence $\varphi\colon M \to \tilde{M}$ such that $\tilde{k}^a \circ \varphi = k^a$ where $k^a, \tilde{k}^a$ are invariants of order $j < q$.*

## 3. Legendre maps in $T_1S^2$ and in $T_1S^3$.

Frenet frames of a Legendre map $\lambda$ in $T_1S^n$ under the action of $PO(n+1,2)$ are called *Lie frames* of $\lambda$. We are here concerned with Legendre maps into $T_1S^n$ and construct their Lie frames in case that $n=2$ and $n=3$.

### 3.1 Lie frames of Legendre maps in $T_1S^2$ under $PO(3,2)$.

**Theorem 3.1.** *Let $\lambda\colon M^1 \to T_1S^2$ be a Legendre map which is induced by an embedded curve $f\colon M^1 \to S^2$ with a field of unit normals $\xi\colon M^1 \to S^2$. Let $\Omega$ be the Maurer-Cartan form on $PO(3,2)$ and $ds$ a coframe field on $M$. Then for $\lambda$ we can construct a Lie frame $u\colon M^1 \to PO(3,2)$ of one of the distinct three types below:*

(1) *type(a): $u^*\Omega = X_a ds$,*
(2) *type(b): $u^*\Omega = X_b ds$,*
(3) *type(c): $u^*\Omega = X_c ds$,*

*where*

$$X_a = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_b = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_c = \begin{pmatrix} 0 & k & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2}k & 0 & 0 & 1 & 0 \\ 0 & -1 & \frac{3}{2}k & 0 & -1 \\ 1 & 0 & 0 & -k & 0 \end{pmatrix}$$

*and $k$ is a smooth function on $M^1$.*

For the proof of Theorem 3.1, See [YY1].

*Remark.* We put the conditon that $u^*\Omega = X ds$ in another way as follows:
We can take a Lie frame $s \to (\, Y_1(s) \quad Y_2(s) \quad Y_3(s) \quad Y_4(s) \quad Y_5(s)\,) \in O(n+1,2)$ such that $(\, \frac{Y_1}{ds} \quad \frac{Y_2}{ds} \quad \frac{Y_3}{ds} \quad \frac{Y_4}{ds} \quad \frac{Y_5}{ds}\,) = (\, Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5\,) X$.

For a curve of type(c), we can take a Lie frame $(Y_1, Y_2, Y_3, Y_4, Y_5)$ that satisfies the following Frenet's formula:

$$\begin{aligned}
\frac{dY_1}{ds} &= Y_2 + \frac{3}{2}kY_3 + Y_5 \\
\frac{dY_2}{ds} &= kY_1 - Y_4 \\
\frac{dY_3}{ds} &= Y_1 + \frac{3}{2}kY_4 \\
\frac{dY_4}{ds} &= Y_3 - kY_5 \\
\frac{dY_5}{ds} &= -Y_4,
\end{aligned}$$

where $k$ is a smooth function on $M^1$. The Lie frame above agrees with the frame which obtained by S. Sasaki and T. Suguri, who examined only general curves ([SS]).

Let us consider the classification of curves obtained in Theorem 3.1 in view of curvature circles. The curvature circle [K] of a Legende map in $T_1S^2$ of type(a), (b) or (c) is equal to $[Y_5]$.

**Corollary 3.2.** *A Legendre map of type(a) is an oriented circle.*

We consider curves of type(b) ("degenerate curves"). For a curve of type(b) we can take the following Lie frame:

$$\begin{aligned}
(\, Y_1(s) \quad Y_2(s) \quad Y_3(s) \quad Y_4(s) \quad Y_5(s)\,) &= \begin{pmatrix} 1 & 0 & s & \frac{s^2}{2} & -\frac{s^3}{6} \\ s & 1 & \frac{s^2}{2} & \frac{s^3}{6} & -\frac{s^4}{24} \\ 0 & 0 & 1 & s & -\frac{s^2}{2} \\ 0 & 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \exp sX_b.
\end{aligned}$$

We describe $\exp sX_b(o)$ in $T_1S^2$ as follows:

$$\begin{aligned}
s &\to \exp sX_b(o) = (f(s), \xi(s)) \in T_1S^2 \\
f(s) &= \frac{1}{\Delta}(-s^6 + 36s^4 - 144s^2 + 576,\ 12s^5 + 576s,\ -96s^3) \\
\xi(s) &= \frac{1}{\Delta}(-192s^3,\ -24s^4 + 288s^2,\ -s^6 - 36s^4 + 144s^2 + 576),
\end{aligned}$$

where $\Delta = s^6 + 36s^4 + 144s^2 + 576$.

We mention the procedure for distinguishing the type of a given curve by the curvature circle. Let $s \to \lambda(s)$ be a curve in $T_1S^2$ and $[K(s)]$ the curvature circle of $\lambda(s)$. When $\frac{dK}{ds}=0$, $\lambda$ is an oriented circle. If $\frac{dK}{ds}\neq 0$ we examine the second differential $\frac{d^2K}{ds^2}$. We let $\frac{d^2K}{ds^2}=kK-Y$. If $k\neq 0$ then $\lambda$ is a general curve with $k\neq 0$. If $k=0$ we examine the fifth differential $\frac{d^5K}{ds^5}$. When $\frac{d^5K}{ds^5}=0$, $\lambda$ is a degenerate curve. When $\frac{d^5K}{ds^5}\neq 0$, $\lambda$ is a general curve with $k=0$.

Finally we obtain the necessary and sufficient condition that two curves are Lie equivalent by virtue of Theorem 2.2 and Theorem 3.1.

**Corollary 3.3.**

(1) *All oriented circles in* $T_1S^2$ *are Lie equivalent.*

(2) *All "degenerate" curves of type(b) in* $T_1S^2$ *are Lie equivalent.*

(3) *Let* $\lambda\colon M\to T_1S^2$, $\tilde\lambda\colon \tilde M\to T_1S^2$ *be smooth curves of type(c), and* $k,\tilde k$ *the smooth functions which are defined in Theorem 3.1 with respect to* $\lambda,\tilde\lambda$ *respectively. Curves* $\lambda$ *and* $\tilde\lambda$ *are Lie equivalent if and only if there exists a one-to-one correspondence* $\varphi\colon M\to\tilde M$ *such that* $k=\varphi^*\tilde k$.

### 3.2 Lie frames of Legendre maps in $T_1S^3$ under $PO(4,2)$.

**Theorem 3.4.** *Let* $\lambda\colon M^2\to T_1S^3$ *be a Legendre map which is induced by an embedded oriented surface* $f\colon M^2\to S^3$ *with a field of unit normals* $\xi\colon M^2\to S^3$. *Let* $\Omega$ *be the Maurer-Cartan form on* $PO(4,2)$ *and* $\phi_1,\phi_2$ *coframe fields on* $M^2$. *Then for* $\lambda$ *we can construct a Lie frame* $u\colon M^1\to PO(3,2)$ *of one of the distinct five types below:*

(1) *type(a):*

$$u^*\Omega=\begin{pmatrix}0&0&\phi_1&\phi_2&0&0\\0&0&0&0&0&0\\0&0&0&0&\phi_1&0\\0&0&0&0&\phi_2&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

(2) *type(b):*

$$u^*\Omega=\begin{pmatrix}0&0&\phi_1&\phi_2&0&0\\0&0&\phi_1&-\phi_2&0&0\\0&0&0&0&\phi_1&\phi_1\\0&0&0&0&\phi_2&-\phi_2\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

(3) *type(c):*

$$u^*\Omega=\begin{pmatrix}\alpha&\delta&0\\\beta&0&{}^t\delta\\\gamma&{}^t\beta&-{}^t\alpha\end{pmatrix},$$

$$\alpha=\begin{pmatrix}-3k^1\phi_1-3k^2\phi_2&k^1\phi_1+k^2\phi_2\\(k^1+1)\phi_1+k^2\phi_2&(-3k^1+1)\phi_1-3k^2\phi_2\end{pmatrix},$$

$$\beta=\begin{pmatrix}k^1\phi_1+(k^2-k^3)\phi_2&k^1\phi_1+(k^2-k^3)\phi_2\\2k^3\phi_1+k^4\phi_2&-2k^3\phi_1-k^4\phi_2\end{pmatrix},$$

$$\gamma=\begin{pmatrix}0&k^4\phi_1\\-k^4\phi_1&0\end{pmatrix},\quad \delta=\begin{pmatrix}\phi_1&\phi_2\\\phi_1&-\phi_2\end{pmatrix}.$$

*where* $k^i$ $(i=1,2,3,4)$ *are smooth functions on* $M^2$ *that satisfy the following integrability conditions:*

$$\begin{aligned}
&k^2+6k^1k^3+k^3_1=0,\\
&12k^2k^3-8k^1k^4+(2k^3_2-k^4_1)=0,\\
&8k^2k^4+k^4_2=0,\\
&2k^1k^2+k^3-k^1_2+k^2_1=0,
\end{aligned}$$

*and* $k^i_j$ $(i=1,2,3,4,\ j=1,2)$ *are smooth functions on* $M^2$ *such that* $dk^i=k^i_1\phi_1+k^i_2\phi_2$,

(4) *type(d):*

$$\begin{aligned}
u^*\Omega&=\begin{pmatrix}\alpha&\delta&0\\ \beta&0&{}^t\delta\\ \gamma&{}^t\beta&-{}^t\alpha\end{pmatrix},\\
\alpha&=\begin{pmatrix}3k^1\phi_1+3k^2\phi_2&k^1\phi_1+k^2\phi_2\\ k^1\phi_1+(k^2+1)\phi_2&3k^1\phi_1+(3k^2-1)\phi_2\end{pmatrix},\\
\beta&=\begin{pmatrix}k^4\phi_1-2k^3\phi_2&k^4\phi_1-2k^3\phi_2\\ (k^1+k^3)\phi_1+k^2\phi_2&(-k^1-k^3)\phi_1-k^2\phi_2\end{pmatrix},\\
\gamma&=\begin{pmatrix}0&k^4\phi_2\\ -k^4\phi_2&0\end{pmatrix},\quad \delta=\begin{pmatrix}\phi_1&\phi_2\\ \phi_1&-\phi_2\end{pmatrix}.
\end{aligned}$$

*where* $k^i$ $(i=1,2,3,4)$ *are smooth functions on* $M^2$ *that satisfy the following integrability conditions:*

$$\begin{aligned}
&k^1-6k^2k^3+k^3_2=0,\\
&12k^1k^3+8k^2k^4+(-2k^3_1-k^4_2)=0,\\
&8k^1k^4-k^4_1=0,\\
&2k^1k^2-k^3-k^1_2+k^2_1=0,
\end{aligned}$$

*and* $k^i_j$ $(i=1,2,3,4,\ j=1,2)$ *are smooth functions on* $M^2$ *such that* $dk^i=k^i_1\phi_1+k^i_2\phi_2$,

(5) *type(e):*

$$\begin{aligned}
u^*\Omega&=\begin{pmatrix}\alpha&\delta&0\\ \beta&0&{}^t\delta\\ \gamma&{}^t\beta&-{}^t\alpha\end{pmatrix},\\
\alpha&=\begin{pmatrix}(3k^1+1)\phi_1-(3k^2+1)\phi_2&k^1\phi_1+k^2\phi_2\\ (k^1+1)\phi_1+(k^2+1)\phi_2&(3k^1+2)\phi_1-(3k^2+2)\phi_2\end{pmatrix},\\
\beta&=\begin{pmatrix}k^5\phi_1+k^3\phi_2&k^5\phi_1+k^3\phi_2\\ k^4\phi_1+k^6\phi_2&-k^4\phi_1-k^6\phi_2\end{pmatrix},\\
\gamma&=\begin{pmatrix}0&k^6\phi_1+k^5\phi_2\\ -k^6\phi_1-k^5\phi_2&0\end{pmatrix},\quad \delta=\begin{pmatrix}\phi_1&\phi_2\\ \phi_1&-\phi_2\end{pmatrix}.
\end{aligned}$$

*where $k^i$ ($i = 1, \ldots, 6$) are smooth functions on $M^2$ that satisfy the following integrability conditions:*

$$\begin{aligned}
&k^3 = (k^1 + k^2)^2 + 2k^1_2 + k^2_1,\\
&k^4 = (k^1 + k^2)^2 - k^1_2 + 2k^2_1,\\
&6k^1k^3 + 4k^2k^5 + 3k^3 + 2k^5 - k^3_1 + k^5_2 = 0,\\
&4k^1k^6 + 6k^2k^4 + 3k^4 + 2k^6 + k^4_2 - k^6_1 = 0\\
&4k^1k^5 + 4k^2k^6 + 2k^5 + 2k^6 + k^5_1 - k^6_2 = 0,
\end{aligned}$$

*and $k^i_j$ ($i = 1, \ldots, 6$, $j = 1, 2$) are smooth functions on $M^2$ such that $dk^i = k^i_1\phi_1 + k^i_2\phi_2$.*

For the proof of Theorem 3.4, See [YY2].

Let us consider the classification of surfaces obtained in Theorem 3.4 in view of curvature spheres. We begin considering surfaces of type(a). The curvature sphere $[K]$ of $\lambda : M^2 \to T_1S^3$ of type(a) is equal to $[Y_6]$ which has maltiplicity 2. From Thorem 4.1 (1), we see that $dY_6 = 0$. Thus $[K]$ is constant, that is, $\lambda$ is an oriented sphere.

*Remark.* If $\mathcal{S}$ is a cuvature surface of dimention $m > 1$ in a Legendre submanifold of $T_1S^n$, then the corresponding curvature sphere is constant along $\mathcal{S}$. (See [P, Proposition 2].) Thus we see that $\lambda$ is an oriented sphere. This fact, moreover, shows that we have only to check the Dupin condition along curvature surfaces with dimension one.

Next we consider surfaces of type(b), (c), (d) or (e). The curvature spheres of $\lambda$ of type(b), (c), (d) or (e) are

$$[K_1(p)] = [Y_5(p) + Y_6(p)], \quad [K_2(p)] = [-Y_5(p) + Y_6(p)].$$

Let $X_1, X_2$ are the principal vectors corresponding to $[K_1], [K_2]$ respectively, that is, $X_1, X_2$ are vectors in $T_pM$ such that $\phi_1(X_1) = 0,\ \phi_2(X_2) = 0$.

If $\lambda$ is of type(b), then

$$dK_1 = 2\phi_1Y_3, \quad dK_2 = -\phi_2Y_4,$$

and hence $dK_1(X_1) = dK_2(X_2) = 0$. Thus along each curvature surface the corresponding curvature sphere is constant, that is, $\lambda$ is a cyclide of Dupin.

Let $\lambda$ be of type(c). We change the functions $k^1,\ k^2$ in Theorem 3.4 (3) so that $h^1 = 4k^1,\ h^2 = 2k^2$. Then

$$\begin{aligned}
dK_1 &= 2\{\phi_1Y_3 + (\frac{h^1-2}{4}\phi_1 + \frac{h^2}{2}\phi_2)Y_5 + (\frac{h^1-2}{4}\phi_1 + \frac{h^2}{2}\phi_2)Y_6\}\\
dK_2 &= 2\{-\phi_2Y_4 + (-\frac{h^1+1}{2}\phi_1 - h^2\phi_2)Y_5 + (\frac{h^1-1}{2}\phi_1 + h^2\phi_2)Y_6\}.
\end{aligned}$$

Let $\lambda$ be of type(d). We change the functions $k^1$, $k^2$ in Theorem 3.4 (4) so that $h^1 = -2k^1$, $h^2 = -4k^2$. Then

$$dK_1 = 2\{\phi_1 Y_3 + (h^1\phi_1 + \frac{h^2-1}{2}\phi_2)Y_5 + (h^1\phi_1 + \frac{h^2+1}{2}\phi_2)Y_6\}$$
$$dK_2 = 2\{-\phi_2 Y_4 + (-\frac{h^1}{2}\phi_1 - \frac{h^2+2}{4}\phi_2)Y_5 + (\frac{h^1}{2}\phi_1 + \frac{h^2+2}{4}\phi_2)Y_6\}.$$

Thus if $\lambda$ is of type(c) and $k^2 = h^2 \equiv 0$,then only $[K_1]$ is constant along $X_1$. If $\lambda$ is of type(d) and $k^1 = h^1 \equiv 0$, then only $[K_2]$ is constant along $X_2$; such $\lambda$ is classically called a canal surface.

**Corollary 3.5.**

(1) *A Legendre map of type(a) is a oriented hypershpere.*
(2) *A Legendre map of type(b) is a cyclide of Dupin.*
(3) *A Legendre map of type(c) (or type(d)) is a canal surface if the function $k^2 \equiv 0$ ($k^1 \equiv 0$), where $k^2$ ($k^1$) is the function defined in Theorem 3.4 (3) ((4)) .*

We can distinguish the type of a given Legendre map $\lambda : M^2 \to T_1S^3$ which has two cuvature spheres $[K_1], [K_2]$ with maltiplicity 1 in the following way. Set

$$dK_1 = 2\phi_1 Y_3 + (A\phi_1 + B\phi_2)K_1 + C\phi_2 K_2,$$
$$dK_2 = -2\phi_2 Y_4 + D\phi_1 K_1 + (E\phi_1 + F\phi_2)K_2,$$

where $A, B, C, D, E, F$ are some smooth functions of $M$. If $C = 0$, $D = 0$, then $\lambda$ is of type(b); i.e. a cyclide of Dupin. If $C = 0$, $D \neq 0$ ($C \neq 0$, $D = 0$), then $\lambda$ is of type(c) (type(d)) ; moreover if $B = 0$ ($E = 0$), then $\lambda$ is a canal surface. If $C \neq 0$, $D \neq 0$, then $\lambda$ is of type(e).

Finally we obtain the necessary and sufficient condition that two surfaces are Lie equivalent by virtue of Theorem 2.2 and Theorem 3.4:

**Corollary 3.6.**

(1) *All oriented spheres in $T_1S^3$ are Lie equivalent.*
(2) *All cyclides of Dupin in $T_1S^3$ are Lie equivalent.*
(3) *Let $\lambda$: $M \to T_1S^2$, $\tilde{\lambda}$: $\tilde{M} \to T_1S^2$ be smooth surfaces of type(c), (d) or (e). Let $k^i, \tilde{k}^i$ be the smooth functions which are defined in Theorem 3.4 (3), (4) or (5) with respect to $\lambda, \tilde{\lambda}$ respectively. Two surfaces $\lambda$ and $\tilde{\lambda}$ are Lie equivalent if and only if there exists a one-to-one correspondence $\varphi$: $M \to \tilde{M}$ such that $k^i = \varphi^* \tilde{k}^i$ for all $i$.*

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